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Magnetic Moment and Virtual Dissociation of Nuclear Particle*

By Gentaro ARAKI

(Received Dec. 1, 1945)

Introduction

The strength of the interaction between a nuclear particle and mesons is determined usually from the potential function of the nuclear force.(1) In such a way one has to solve, for example, the deuteron problem. A more direct one may be to examine the magnetic moment of the nuclear particle.

As is well known, Yukawa's theory of meson,(2) as well as Fermi's theory of the nuclear β-decay,(3) gives an unsatisfactory quantitative result to the magnetic moment. It is generally believed that, if this divergence difficulty is disregarded the surplus magnetic moment of the nuclear particle arises from its virtual dissociation emitting charged mesons. The former can accordingly be represented by the degree of the latter.

In the present paper this dissociation probability is estimated from a difference between absolute values of the surplus magnetic moment of the proton and the magnetic moment of the neutron. Its value is found to be 0.113 almost independently of its explicit theoretical expression. This result suggests that the interaction of mesons with a nuclear particle is comparatively weak. The cut-off hypothesis gives the constant of the pseudovector interaction between a nuclear particle and mesons a value $f^2/\hbar c=8.9$, which is irreconcilable with the experiment of the nuclear scattering of mesons (4)

The following calculation is carried out according to the pseudoscalar theory of meson, the constant of the pseudoscalar interaction being omitted. The absolute values of the surplus magnetic moment of the proton and the magnetic moment of the neutron are the same in the Schrödinger approximations. One can find their difference only in the Pauli and higher approximation. In the present calculation all the terms in the Pauli approximation are completely taken into account.(5)

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* The content of this paper was reported at the Symposium of the National Reserch Council of Japan, on Nov. 18, 1944.

(1) E.g. H.A. Bethe, Phys. Rev. 57 (1940), 260, 390.

(2) H. Fröhlich, W. Heitler and N. Kemmer, Proc. Roy. Soc. London (A), 166 (1938), 154.
S. Ueno and H. Kuze, Proc. Phys. Math. Soc. Japan, 24 (1942), 184.
Z. Yamasaki and S. Uma, ibid., 207.
G. Araki, Sci. Pap. I.P.C.R., 42 (1944), 113.

(3) C. F. v. Weizsäcker, ZS. f. Phys., 102 (1936), 572.
M. Fierz, ibid., 104 (1937), 553.

(4) G. Araki, Sci. Pap. I.P.C.R., 40 (1943), 311.

1. Hamiltonian of a nuclear particle in a static magnetic field

The transition from a state *A* to a state *F*, which is different from *A*, of a quantum mechanical system, described by a Hamiltonian *H*, is determined by the following materix-elements:

$$\Big(1-\sum_{I}\frac{|H_{IA}|^2}{(E_I-E_A)^2}+\cdots\Big)\Big(H_{FA}-\sum_{I}\frac{H_{FI}H_{IA}}{E_I-E_A}$$

$$+\sum_{I,I'}\frac{H_{FI'}H_{I'I}H_{IA}}{(E_{I'}-E_A)\,(E_I-E_A)}-\cdots\Big)+\cdots, \qquad \text{.................}(1.\ 1)$$

where all the diagonal elements of unperturbed part of *H* are omitted from the summands, and the sum of such a kind that the donominator of its summand approaches to zero is excluded. E_I and E_A are eigenvalues of a unperturbed Hamiltonian which can be taken arbitrarily if the degree of the approximation is left out of consideration.

The Hamiltonian of a system, which consists of a nuclear particle and charged mesons in a static magnetic field, is given by

$$H=H^{n}+H^{m}+H^{ne}+H^{me}+H^{nm}+H^{nme}, \qquad \text{........................}(1.\ 2)$$

where H^n and H^m are energies of an isolated nuclear particle and isolated mesons respectively, H^{ne} and H^{me} are respectively interactions of a nuclear particle and mesons with a magnetic field, H^{nm} is that between a nuclear particle and mesons, and H^{nme} is a mixed one between a nuclear particle, mesons and the field.

If all the matrix-elements are represented with respect to eigenstates of H^n+H^m, and if there is no meson in the states *A* and *F*, the terms having a factor ef^2 in the expression (1. 1) are

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(5) A calculation in such an approximation was carried out by the present author (G. Araki, l. c.). This result is, however, insufficient for the purpose of the present paper.

$$\left.\begin{aligned}
U^{(1)}_{FA} &= \sum_{I,I'}\frac{H^{nm}_{FI'}\,H^{me}_{I'I}\,H^{nm}_{IA}}{(E_{I'}-E_A)\,(E_I-E_A)} && \text{(a)}\\
U^{(2)}_{FA} &= \sum_{I,II}\frac{H^{nm}_{FI}H^{nm}_{I\,II}H^{me}_{IIA}+H^{me}_{FII}H^{nm}_{II\,I}H^{nm}_{IA}}{(E_{II}-E_A)(E_I-E_A)} && \text{(b)}\\
U^{(3)}_{FA} &= -\sum_{I}\frac{H^{nm}_{FI}H^{nme}_{IA}+H^{nme}_{FI}H^{nm}_{IA}}{E_I-E_A} && \text{(c)}\\
U^{(4)}_{FA} &= \sum_{I,I'}\frac{H^{nm}_{FI'}\,H^{ne}_{I'I}\,H^{nm}_{IA}}{(E_{I'}-E_A)\,(E_I-E_A)} && \text{(d)}\\
U^{(5)}_{FA} &= -H^{ne}_{FA}\sum_{I}\frac{|\,H^{nm}_{IA}\,|^2}{(E_I-E_A)} && \text{(e)}
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(1.\ 3)$$

where e is an elementary positive charge, f is a constant of the pseudovector interaction between a nuclear particle and mesons, I (or I') and II denote the states in which one and two mesons are present respectively. Let $U^{(1)}$, $U^{(2)}$, $U^{(3)}$, $U^{(4)}$ and $U^{(5)}$ be the operators which have respectively the same matrix-elements as (1. 3) (a), (b), (c), (d), and (e), with respect to eigenstates of H^n. Their sum

$$U=U^{(1)}+U^{(2)}+U^{(3)}+U^{(4)}+U^{(5)}\quad\cdots\cdots\cdots\cdots(1.\ 4)$$

represents an additional energy, which arises from the virtual dissociation, of a nuclear particle in a magnetic field.

2. Theoretical result

As will be shown in the later section, if the terms, which vanish when they operate upon a wave function of a rest nuclear particle, are omitted, the operators $U^{(k)}$ ($k=1, 2, 3, 4, 5$) are, in the Pauli approximation, given as follows:

$$\left.\begin{aligned}
U^{(k)} &= \mu^{(k)}\frac{e\hbar}{2Mc}\tau_3\sigma\ \mathrm{rot}\ \mathbf{A}\quad (k=1,\ 2,\ 3) && \text{(a)}\\
U^{(4)} &= \frac{1}{3}W_D\frac{e\hbar}{2Mc}\tau\tau^{\dagger}\sigma\ \mathrm{rot}\ \mathbf{A} && \text{(b)}\\
U^{(5)} &= W_D\frac{e\hbar}{2Mc}\tau^{\dagger}\tau\sigma\ \mathrm{rot}\ \mathbf{A} && \text{(c)}\\
\mu^{(1)} &= \frac{2}{3}\left(\overline{\frac{Mc^2}{E}}-1\right)W_D && \text{(d)}\\
\mu^{(2)} &= \left\{\frac{2}{3}\,\overline{\frac{Mc^2}{E}}-\frac{1}{3}\overline{\left(\frac{\mu c^2}{E}\right)^2}-\frac{1}{3}\right\}W_D && \text{(e)}\\
\mu^{(3)} &= \left\{\frac{4}{3}+2\overline{\left(\frac{\mu c^2}{p}\right)^2}\right\}W_D && \text{(f)}
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(2.1)$$

where **A** is a vector potential of the magnetic field, μ and M are masses of a nuclear particle and a meson respectively, and σ is the Dirac matrix (vector) of the nuclear particle, τ_1, τ_2, τ_3 are the charge operators of the nuclear particle. The neutron state belongs to the eigenvalue $+1$ of τ_3, τ denotes $(\tau_1+i\tau_2)/2$, and $\tau^\dagger$ is its adjoint. $\tau^\dagger\tau$ and $\tau\tau^\dagger$ are the projection operators to the proton and the neutron states respectively. W_D is the dissociation probability of the nuclear particle virtually emitting a charged meson, $\mathbf{p}/c$ and E are respectively the momentum and the energy of the latter, and the bar means the average regarding the dissociation states.

Using the relation $\tau\tau^\dagger-\tau^\dagger\tau=\tau_3$, we have, from (2. 1) and (I. 4), the magnetic energy operator as follows:

$$\left.\begin{aligned} &\mathrm{U}=\frac{e\hbar}{2\mathrm{Mc}}\left(\mu_\mathrm{P}\tau_3+\frac{4}{3}\mathrm{W_D}\,\tau\tau^\dagger\right)\sigma\ \mathrm{rot}\ \mathbf{A}, && \text{(a)}\\ &\mu_\mathrm{P}=\left\{\frac{4}{3}\,\overline{\frac{\mathrm{Mc}^2}{\mathrm{E}}}+2\overline{\left(\frac{\mu c^2}{\mathrm{p}}\right)^2}-\frac{1}{3}\overline{\left(\frac{\mu c^2}{\mathrm{E}}\right)^2}-\frac{2}{3}\right\}\mathrm{W_D} && \text{(b)}\end{aligned}\right\}\quad\ldots(2.\ 2)$$

In this expression μ_P represents the surplus magnetic moment of the proton in nuclear magneton units, for $\tau\tau^\dagger$ is the projection operator to the neutron state. The magnetic moment of the neutron is, for the same reason, given by

$$\mu_\mathrm{N}=-\left(\mu_\mathrm{P}+\frac{4}{3}\mathrm{W_D}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.\ 3)$$

(The exact formulas of μ_P, μ_N and W_D are given in the note added in proof in p. 11.)

3. Comparison with experiment

The experimental values of μ_P and μ_N are given by[6]

$$\left.\begin{aligned}\mu_\mathrm{P}&=+1.785\\ \mu_\mathrm{N}&=-1.935\end{aligned}\right\}\ \text{(in nuclear magnetons)}\ \ldots\ldots\ldots\ldots\ldots\ldots(3.\ 1)$$

The theoretical result (2. 3) agrees qualitatively with these experimental values, since W_D must be positive and small compared with unity. The adequacy of our approximation can be seen from the following consideration, if we assume the correctness of the general theory.

From (2. 1) one finds that $U^{(3)}$, $U^{(4)}$ and $U^{(5)}$ vanish in the Schrödinger approximation. The difference between absolute values of the surplus

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(6) L. W. Alvarez and F. Bloch, Phys. Rev., 57 (1940), 111

magnetic moment of the proton and the magnetic moment of the neutron arises from $U^{(4)}$ and $U^{(5)}$, which can differ from zero only in the Pauli and higher approximations. Here the Schrödinger approximation means the terms of degree zero of the power series in $1/M$ and the Pauli approximation the zeroth and first degree terms.

The ratio of the difference to the average, of the absolute values of μ_P and μ_N, is, from (3. 1), given by

$$\frac{|\mu_N|-\mu_P}{(|\mu_N|+\mu_P)/2}=\frac{0.150}{1.860}=0.08 \quad \dots\dots\dots\dots\dots\dots (3.\ 2)$$

The smallness of this ratio suggests that the power series in $1/M$ converges rapidly so that the Pauli approximation is satisfactory. The value of the dissociation probability of the nuclear particle is obtained from (2. 3) and (3. 1):

$$W_D=-\frac{3}{4}(\mu_N+\mu_P)=0.113 \dots\dots\dots\dots\dots\dots (3.\ 3)$$

The smallness of this probability implies that the interaction between a nuclear particle and mesons is weak so that our perturbation method has a significance.

We shall next consider the distribution, with respect to the momentum p/c of a virtual meson, of the dissociation probability. When the maximum point p_m of this distribution is near μc^2, the equation (2. 2) gives the value 3.6 to the average of $(\mu c^2/p)^2$. In case $p_m \ll \mu c^2$ this value becomes 1.8, and in case $p_m \gg \mu c^2$ the average value of Mc^2/E is nearly equal to 12, by the same equation. The maximum of the distribution may therefore be at the point slightly less than μc^2. We may estimate a decrease, due to the above mentioned virtual dissociation, of a self-energy of a nuclear particle at about 0.1 μc^2, for it is given by EW_D.

4. Cut-off hypothesis

We have compared the theory with the experiment disregarding the explicit expression of the dissociation probability in the preceding section. We shall now consider this explicit expression. In the present theory, this includes the divergent integrals. If we adopt the usual cut-off hypothesis, μ_P and W_D, as the functions of the cut-off momentum $x\mu c$, are given by

$$\left.\begin{aligned} \frac{\mu_P}{W_D} &= \Phi(x) - \frac{2}{3} && \text{(a)} \\ W_D &= \frac{f^2}{\hbar c}\,\frac{G(x)}{\pi} && \text{(b)} \end{aligned}\right\} \quad \text{.......................................(4. 1)}$$

where

$$\left.\begin{aligned} \Phi(x) &= \frac{F(x)}{G(x)} && \text{(a)} \\ F(x) &= \frac{4}{3}\,\frac{M}{\mu}\int_0^x \frac{x^4 dx}{(1+x^2)^2} + 2\int_0^x \frac{x^2 dx}{\sqrt{1+x^2}^{\,3}} + \int_0^x \frac{x^4 dx}{\sqrt{1+x^2}^{\,5}} && \text{(b)} \\ G(x) &= \int_0^x \frac{x^4 dx}{\sqrt{1+x^2}^{\,3}} + \frac{\mu}{M}\int_0^x \frac{x^4 dx}{(1+x^2)^2} && \text{(c)} \end{aligned}\right\} \text{(4. 2)}$$

Substituting (3. 1) and (3. 3) into (4. 1) (a) we have the equation for the cut-off point x:

$$\Phi(x) = 16.5 \quad \text{...(4. 3)}$$

This equation is numerically solved, using the value $M = 10\mu$. Its root is $x = 0.8$. The corresponding value of $G(x)$, evaluated by the numerical integration, is equal to 0.04. Inserting this value and (3. 3) in (4. 1) (b), we obtain the value of the interaction constant:

$$\frac{f^2}{\hbar c} = 8.9 \quad \text{...(4. 4)}$$

This value is never reconcilable with Wilson's experiment of a nuclear scattering of mesons.(7) Of course the inadequacy of the cut-off hypothesis may be evident from the outset, and the discrepancy must closely be related to the general difficulty which prevent us from any consistent treatment of the quantum mechanics of the field. Nevertheless a qualitative agreement of the theory with the experiment is seen in the previous section.

5. Interaction energy

Consider a quantum mechanical system consisting of a nuclear particle and mesons in a static magnetic field described by a solenoidal vector potential $\mathbf{A}(\mathbf{x})$. The whole system is assumed to be enclosed in a sufficiently large cube V of an edge length $\sqrt[3]{V}$. The interaction energies in (1. 2) are respectively given by

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(7) J. G. Wilson, Proc. Roy. Soc. London (A), 172 (1939), 517; 174 (1940), 73.
F. L. Code, Phys. Rev., 59 (1941), 229.
G. Araki, Sci. Pap. I.P.C.R, 40 (1943), 311.

$$\left.\begin{aligned}
&\mathrm{H}^{ne}=-\tau^{\dagger}\tau e\rho_1\sigma\mathbf{A} && \text{(a)}\\
&\mathrm{H}^{me}=-e\sum_{\mathbf{p},\mathbf{p}'}\frac{\mathbf{p}\mathbf{A}_{\mathbf{p},\mathbf{p}'}}{\sqrt{\mathrm{EE}'}}\left\{(\mathrm{a}_{\mathbf{p}}^{\dagger}-\mathrm{b}_{-\mathbf{p}})(\mathrm{a}_{\mathbf{p}'}-\mathrm{b}_{-\mathbf{p}'}^{\dagger})-1\right\} && \text{(b)}\\
&\mathrm{H}^{nm}=\frac{f}{\varkappa}\sqrt{\frac{2\pi}{\mathrm{V}}}\sum_{\mathbf{p}}\frac{1}{\sqrt{\mathrm{E}}}(\mathbf{p}\sigma-\mathrm{E}\rho_1)(\tau\mathrm{a}_{\mathbf{p}}^{\dagger}+\tau^{\dagger}\mathrm{b}_{\mathbf{p}}^{\dagger})e^{-i\mathbf{p}\mathbf{x}/\hbar c}+\text{its adjoint} && \text{(c)}\\
&\mathrm{H}^{nme}=\frac{ef}{\varkappa}\sigma\mathbf{A}\sqrt{\frac{2\pi}{\mathrm{V}}}\sum_{\mathbf{p}}\frac{1}{\sqrt{\mathrm{E}}}(-\tau\mathrm{a}_{\mathbf{p}}^{\dagger}+\tau^{\dagger}\mathrm{b}_{\mathbf{p}}^{\dagger})e^{-i\mathbf{p}\mathbf{x}/\hbar c}+\text{its adjoint} && \text{(d)}
\end{aligned}\right\}\quad(5.\ 1)$$

where

$$\left.\begin{aligned}
\mathrm{E}&=\sqrt{\mu^2c^4+\mathbf{p}^2}\\
\mathrm{E}'&=\sqrt{\mu^2c^4+\mathbf{p}'^2}
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots(5.\ 2)$$

and $\mathbf{A}_{\mathbf{p}\mathbf{p}'}$ is a matrix element of $\mathbf{A}$ between the states of momenta $\mathbf{p}$ and $\mathbf{p}'$ of the plane wave. The other notations used in these expressions are all usual.(8)

6. Calculation of magnetic energy operator

We shall now explain the method of the calculation of the matrix-elements (1. 3). Let the momenta of the nuclear particle in the states A and F be denoted by $\mathbf{p}_0/c$ and $\mathbf{p}_1/c$ respectively. We expand the expressions of (1. 3) in powers of p_0. The terms of the n-th power of p_0 ($n=1$ 2, ...) in this series correspond to that of the n-th power of the component of ∇ in the operator U. These terms can be omitted since they vanish when U operates upon a wave function of a rest nuclear particle. This omission is equivalent to put $p_0=0$ in the expressions of (1. 3) from the outset.

The summation over the intermediate states with respect to the spin and the charge can be carried out by the Perseval identity according to the well known method of Casimir. If the spin and the charge states in A and F are left arbitrary, the matrix elements $U_{FA}^{(k)}$ ($k=1, 2, 3, 4$) are given by

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(8) The explanation is given in G. Araki, Sci. Pap. I.P.C.R., 42 (1944), 113.

$$\left.\begin{aligned}
U^{(1)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\frac{(\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})(\mathbf{p}\sigma+\mathbf{p}_1\sigma-E'\rho_1)\Omega'\Omega(\mathbf{p}\sigma-E\rho_1)}{EE'(2Mc^2E-\mu^2c^4)(2Mc^2E'-\mu^2c^4-2\mathbf{p}\mathbf{p}_1-\mathbf{p}_1^2)} && \text{(a)}\\
U^{(2)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}(\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})\left\{\frac{(\mathbf{p}\sigma+\mathbf{p}_1\sigma+E'\rho_1)\Omega(\mathbf{p}\sigma-E\rho_1)}{EE'(E+E'+W-Mc^2)(2Mc^2E-\mu^2c^4)}\right.\\
&\qquad\left.+\frac{(\mathbf{p}\sigma-E\rho_1)\Omega_1(\mathbf{p}\sigma-\mathbf{p}_1\sigma+E''\rho_1)}{EE''(E+E'')(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)}\right\} && \text{(b)}\\
U^{(3)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\left\{\frac{(\sigma\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})\Omega(\mathbf{p}\sigma-E\rho_1)}{E(2Mc^2E-\mu^2c^4)}+\frac{(\mathbf{p}\sigma-E\rho_1)\Omega_1(\sigma\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})}{E(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)}\right\} && \text{(c)}\\
U^{(4)}_{\mathbf{p}_1\mathbf{p}_0} &= -\tau\tau^\dagger e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\frac{(\mathbf{p}\sigma-E\rho_1)\Omega_1(\rho_1\sigma\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})\Omega(\mathbf{p}\sigma-E\rho_1)}{E(2Mc^2E-\mu^2c^4)(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)} && \text{(d)}
\end{aligned}\right\}(6.1)$$

where

$$\left.\begin{aligned}
\Omega &= -\mathbf{p}\sigma\rho_1+Mc^2\rho_3+Mc^2-E\\
\Omega' &= -\mathbf{p}\sigma\rho_1+Mc^2\rho_3+Mc^2-E'\\
\Omega_1 &= (\mathbf{p}_1-\mathbf{p})\sigma\rho_1+Mc^2\rho_3+Mc^2-E
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad (6.\ 2)$$

and

$$\left.\begin{aligned}
E &= \sqrt{\mu^2c^4+\mathbf{p}^2}\\
E' &= \sqrt{E^2+2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2}\\
E'' &= \sqrt{E^2-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2}\\
W &= \sqrt{M^2c^4+\mathbf{p}_1^2}
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad (6.\ 3)$$

It is sufficient for the present purpose to consider the terms which have an effect on the first degree terms of $1/M$ and e in the final result. We can consequently put $\rho_3=1$ and $\rho_1\rho_3=-i\rho_2=\rho_1$. An isolated ρ_1 is the first degree term of $1/M$. The expressions (6. 1) are thus reduced to

$$\begin{aligned}
U^{(1)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\frac{(\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})2Mc^2\{(2Mc^2-E')(\mathbf{p}\mathbf{p}_1+i\mathbf{p}[\sigma\mathbf{p}_1])+2Mc^2(\mathbf{p}^2-E'\mathbf{p}\sigma\rho_1)\}}{EE'(2Mc^2E-\mu^2c^4)(2Mc^2E'-\mu^2c^4-2\mathbf{p}\mathbf{p}_1-\mathbf{p}_1^2)} \quad \text{(a)}\\
U^{(2)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}(\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})\left\{\frac{2Mc^2(i\mathbf{p}[\sigma\mathbf{p}_1]+E'\mathbf{p}\sigma\rho_1+\mathbf{p}\mathbf{p}_1+\mathbf{p}^2)+\mu^2c^4E'}{EE'(E+E'+W-Mc^2)(2Mc^2E-\mu^2c^4)}\right.\\
&\left.+\frac{(2Mc^2-E-E')i\mathbf{p}[\sigma\mathbf{p}_1]+2Mc^2E(\mathbf{p}_1-\mathbf{p})\sigma\rho_1+2Mc^2\mathbf{p}^2-(2Mc^2+E-E')\mathbf{p}\mathbf{p}_1+E\mathbf{p}_1^2+\mu^2c^4E'}{EE''(E+E'')(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)}\right\} \quad \text{(b)}\\
U^{(3)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau_3 e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\left[\frac{\{2Mc^2(E\rho_1-\mathbf{p}\sigma)+E\mathbf{p}_1\sigma\}(\sigma\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})}{E(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)}-\frac{2Mc^2(\sigma\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})(\mathbf{p}\sigma)}{E(2Mc^2E-\mu^2c^4)}\right] \quad \text{(c)}\ (6.4)\\
U^{(4)}_{\mathbf{p}_1\mathbf{p}_0} &= \tau\tau^\dagger e\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_{\mathbf{p}}\\
&\frac{2Mc^2i\{(\mathbf{p}[\sigma\mathbf{p}_1])(\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0})+(\mathbf{p}\mathbf{p}_1)(\mathbf{p}[\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0}\sigma])-(\mathbf{p}\sigma)(\mathbf{p}[\mathbf{p}_1\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0}])+2i\mu^2c^4\mathbf{p}\mathbf{A}_{\mathbf{p}_1\mathbf{p}_0}\}}{E(2Mc^2E-\mu^2c^4)(2Mc^2E-\mu^2c^4-2\mathbf{p}\mathbf{p}_1+\mathbf{p}_1^2)} \quad \text{(d)}
\end{aligned}$$

When these expressions are expanded in powers of $\mathbf{p}_1$, each term of this series is represented by a sum with respect to $\mathbf{p}$. The summands can therefore be replaced by their averages over the direction of $\mathbf{p}$. If we denote such an average by a bar, we have

$$\left.\begin{array}{ll}\overline{\mathbf{pA}}=0 & \text{(a)}\\ \overline{(\mathbf{pA})(\mathbf{pB})}=\frac{1}{3}\,\mathbf{p}^2(\mathbf{AB}) & \text{(b)}\\ \overline{(\mathbf{pA})(\mathbf{pB})(\mathbf{pC})}=0 & \text{(c)}\\ \overline{(\mathbf{pA})(\mathbf{pB})(\mathbf{pC})(\mathbf{pD})}=\frac{1}{15}\,\mathbf{p}^4\{(\mathbf{AB})(\mathbf{CD})+(\mathbf{AC})(\mathbf{BD})+(\mathbf{AD})(\mathbf{BC})\} & \text{(d)}\end{array}\right\}\ (6.5)$$

etc.

Many terms of the series vanish as a result of this average, and three kinds of terms having one of the factors $\mathbf{p}_1\mathbf{A}_{p_1p_0}$, $\sigma[\mathbf{p}_1\mathbf{A}_{p_1p_0}]$ and $\sigma\mathbf{A}_{p_1p_0}$ remain there.

On account of the Hermitain property of the momentum operator and the solenoidal property of the vector potential, the factor of the first kind can be transformed as follows:

$$\mathbf{p}_1\mathbf{A}_{p_1p_0}=\mathbf{p}_0\mathbf{A}_{p_1p_0}+\frac{c\hbar}{i}(\mathrm{div}\ \mathbf{A})_{p_1p_0}=\mathbf{p}_0\mathbf{A}_{p_1p_0}\cdots\cdots\cdots\cdots(6.\ 6)$$

This has a factor $\mathbf{p}_0$. The terms of the first kind are accordingly removed. By the same reason we can put

$$[\mathbf{p}_1\mathbf{A}_{p_1p_0}]=\frac{c\hbar}{i}(\mathrm{rot}\ \mathbf{A})_{p_1p_0}\cdots\cdots\cdots\cdots(6.\ 7)$$

We have now only two kinds of terms having either a factor $\sigma(\mathrm{rot}\ \mathbf{A})_{p_1p_0}$, or $\sigma\mathbf{A}_{p_1p_0}$. The another factors of these terms have either the zeroth or the positive powers of $\mathbf{p}_1^2$. The latter satisfies the following indentity:

$$\left.\begin{array}{ll}\mathbf{p}_1^2\sigma(\mathrm{rot}\ \mathbf{A})_{p_1p_0}=(\mathbf{p}_0\mathbf{p}_1)\sigma(\mathrm{rot}\ \mathbf{A})_{p_1p_0}+\frac{c\hbar}{i}(\mathbf{p}_0\,\mathrm{grad}\,\{\sigma\ \mathrm{rot}\ \mathbf{A}\})_{p_1p_0} & \text{(a)}\\ \mathbf{p}_1^2\sigma\mathbf{A}_{p_1p_0}=(\mathbf{p}_0\mathbf{p}_1)(\sigma\mathbf{A}_{p_1p_0})+\frac{c\hbar}{i}(\mathbf{p}_0\,\mathrm{grad}\,\{\sigma\mathbf{A}\})_{p_1p_0} & \text{(b)}\end{array}\right\}\ (6.\ 8)$$

for the magnetic field is external and its vector potential satisfies the Laplace equation. These terms can therefore be all omitted.

The matrix-elements (6. 4) are thus reduced to

$$\left.\begin{aligned}
\mathrm{U}^{(1)}_{\mathrm{p_1p_0}} &= \tau_3\frac{f^2}{\chi^2}\frac{2\pi}{\mathrm{V}}\sum_{\mathrm{p}}\frac{\mathbf{p}^2}{\mathrm{E}^3}\left\{\left(\frac{2\mathrm{Mc}^2}{3\mathrm{E}}+\frac{2\mu^2\mathrm{c}^4}{3\mathrm{E}^2}-\frac{1}{3}\right)\frac{e\hbar}{2\mathrm{Mc}}\sigma\,(\mathrm{rot}\,\mathbf{A})_{\mathrm{p_1p_0}}-\frac{1}{3}e\rho_1\sigma\mathbf{A}_{\mathrm{p_1p_0}}\right\}\\
\mathrm{U}^{(2)}_{\mathrm{p_1p_0}} &= \tau_3\frac{f^2}{\chi^2}\frac{2\pi}{\mathrm{V}}\sum_{\mathrm{p}}\frac{\mathbf{p}^2}{\mathrm{E}^3}\left(\frac{2\mathrm{Mc}^2}{3\mathrm{E}}+\frac{\mu^2\mathrm{c}^4}{3\mathrm{E}^2}-\frac{1}{3}\right)\frac{e\hbar}{2\mathrm{Mc}}\sigma(\mathrm{rot}\,\mathbf{A})_{\mathrm{p_1p_0}}\\
\mathrm{U}^{(3)}_{\mathrm{p_1p_0}} &= \tau_3\frac{f^2}{\chi^2}\frac{2\pi}{\mathrm{V}}\sum_{\mathrm{p}}\left\{\left(\frac{1}{3\mathrm{E}}+\frac{2\mu^2\mathrm{c}^4}{3\mathrm{E}^3}\right)\frac{e\hbar}{2\mathrm{Mc}}\sigma\,(\mathrm{rot}\,\mathbf{A})_{\mathrm{p_1p_0}}+\frac{1}{\mathrm{E}}e\rho_1\sigma\mathbf{A}_{\mathrm{p_1p_0}}\right\}\\
\mathrm{U}^{(4)}_{\mathrm{p_1p_0}} &= \tau\tau^{\dagger}\frac{f^2}{\chi^2}\frac{2\pi}{\mathrm{V}}\sum_{\mathrm{p}}\frac{\mathbf{p}^2}{\mathrm{E}^3}\cdot\frac{1}{3}\cdot\frac{e\hbar}{2\mathrm{Mc}}\sigma(\mathrm{rot}\,\mathbf{A})_{\mathrm{p_1p_0}}
\end{aligned}\right. \qquad (6.\ 9)$$

Finally the matrix-elements $\mathrm{U}^{(5)}_{\mathrm{FA}}$ is

$$\mathrm{U}^{(5)}_{\mathrm{p_1p_0}}=\tau^{\dagger}\tau\mathrm{W_D}e\rho_1\sigma\mathbf{A}_{\mathrm{p_1p_0}} \quad \cdots\cdots\cdots\cdots(6.\ 10)$$

where

$$\mathrm{W_D}=\sum_{\mathrm{I}}\frac{|\mathrm{H}^{\mathrm{int}}_{\mathrm{IA}}|^2}{(\mathrm{E_I}-\mathrm{E_A})^2}\quad\cdots\cdots\cdots\cdots(6.\ 11)$$

is the probability of the dissociation of a nuclear particle virtually emitting a charged meson. By the above mentioned method the latter is reduced to

$$\mathrm{W_D}=\frac{f^2}{\chi^2}\frac{2\pi}{\mathrm{V}}\sum_{\mathrm{p}}\frac{\mathbf{p}^2}{\mathrm{E}^3}\left(1+\frac{\mu^2\mathrm{c}^4}{\mathrm{Mc}^2\mathrm{E}}\right)\quad\cdots\cdots\cdots\cdots(6.\ 12)$$

From (6. 9), (6. 10), and (6, 12) the operators $U^{(k)}$ ($k=1, 2, 3, 4, 5$) are represented in terms of W_D as follows:

$$\left.\begin{aligned}
\mathrm{U}^{(1)} &= \left\{\frac{2}{3}\overline{\frac{\mathrm{Mc}^2}{\mathrm{E}}}-\frac{1}{3}\right\}\mathrm{W_D}\frac{e\hbar}{2\mathrm{Mc}}\tau_3\sigma\,\mathrm{rot}\,\mathbf{A}-\frac{1}{3}\mathrm{W_D}\tau_3e\rho_1\sigma\mathbf{A} && \text{(a)}\\
\mathrm{U}^{(2)} &= \left\{\frac{2}{3}\overline{\frac{\mathrm{Mc}^2}{\mathrm{E}}}-\frac{1}{3}\overline{\left(\frac{\mu\mathrm{c}^2}{\mathrm{E}}\right)^2}-\frac{1}{3}\right\}\mathrm{W_D}\frac{e\hbar}{2\mathrm{Mc}}\tau_3\sigma\,\mathrm{rot}\,\mathbf{A} && \text{(b)}\\
\mathrm{U}^{(3)} &= \left\{\frac{1}{3}\overline{\left(\frac{\mathrm{E}}{\mathrm{p}}\right)^2}+\frac{2}{3}\overline{\left(\frac{\mu\mathrm{c}^2}{\mathrm{p}}\right)^2}\right\}\mathrm{W_D}\frac{e\hbar}{2\mathrm{Mc}}\tau_3\sigma\,\mathrm{rot}\,\mathbf{A}+\overline{\left(\frac{\mathrm{E}}{\mathrm{p}}\right)^2}\mathrm{W_D}\tau_3e\rho_1\sigma\mathbf{A} && \text{(c)}\\
\mathrm{U}^{(4)} &= \frac{1}{3}\mathrm{W_D}\frac{e\hbar}{2\mathrm{Mc}}\tau\tau^{\dagger}\sigma\,\mathrm{rot}\,\mathbf{A} && \text{(d)}\\
\mathrm{U}^{(5)} &= \mathrm{W_D}\tau^{\dagger}\tau e\rho_1\sigma\mathbf{A} && \text{(e)}
\end{aligned}\right\}\ (6.\ 13)$$

where p/c and E are respectively the momentum and the energy of a virtual meson, and the bar denotes an average over the dissociation states. In the Pauli approximation the expressions of (6. 13) are just the same with those of (2. 1).

Note added in proof. Prof. M. Kobayasi points out that the higher degree terms in $1/M$ may have some effective contribution. The exact calculation shows, however, that this is not the case. ·The result of the exact calculation completely reserving all the terms proportional to f is as follows:

$$\mu_p = \frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_p \frac{1}{E^3\left(1-\frac{\mu^2c^2}{2ME}\right)^2}\left\{\frac{4}{3}\frac{M}{E}p^2 - \frac{2}{3}p^2 + 2\mu^2 - \frac{\mu^2c^4}{3E^2}p^2\right.$$

$$\left. -\frac{\mu}{3M}\left(6E\mu c^2 - \frac{\mu c^2}{E}p^2\right) + \frac{\mu^2}{12M^2}\left(2+\frac{\mu^2c^4}{E^2}\right)\mu^2c^4\right\}$$

$$\mu_N + \mu_P = -\frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_p \frac{1}{E^3\left(1-\frac{\mu^2c^2}{2ME}\right)^2}\left(\frac{4}{3}p^2 + \frac{\mu}{M}E\mu c^2\right)$$

$$W_D = \frac{f^2}{\varkappa^2}\frac{2\pi}{V}\sum_p \frac{1}{E^3\left(1-\frac{\mu^2c^2}{2ME}\right)^2}\left(p^2 + \frac{\mu^2}{4M^2}\mu^2c^4\right)$$

This result just agrees with (2. 2) and (2. 3) up to the first degree in $1/M$. The higher degree terms have no effective contribution, whether the cut-off point is nearly equal to or greater than μc^2.

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On the Generalized Transformation Functions*

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1. Introduction

Dirac⁽¹⁾ has anticipated, in the quantum theory of fields, an existance of the generalized transformation function (g. t. f.) whose square of the modulus can be interpreted as giving the relative a priori probability of specified results being obtained when observations are made of the field variables at all points on any closed boundary surface (three-dimentional) of a region in four-dimentional space-time.

In connexion with the well known difficulties existing in the theory of the elementary particles, those associated with an infinite self-energy of an elementary particle and the diverging cross-sections of the elementary processes etc., Yukawa⁽²⁾ pointed out again the importance of the conception of Dirac's g.t.f. and has done closer considerations about it.

Recently Tomonaga⁽³⁾ has shown that by means of his new formulation of field quantum mechanics, the so-called "super-many-time-theory" being a generalization of Dirac's "many-time-theory" in the field theory, the g.t.f. can exist in the special cases concerning the regions of space-time bounded by two (three-dimentional) surfaces, each extending to infinity in the space directions and lying entirely outside any light-cone having its vertex on the surfaces. These two surfaces may be separated completely or partly in finite numbers of regions.

Then it would seem desirable to take up the questions of whether or not we can really expect the existence of the g.t.f.'s for regions bounded by any surfaces in space-time as Dirac expected, and that if they are expected, in what meanings they can exist.

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* This paper was read before the symposium of the National Research Council of Japan held on Nov. 18, 1944, the printing was however delayed owing to the circumstances in the war.

(1) P. A. M. Dirac: Phys. Zs. d. Sowj., 3 (1933), 64.

(2) H. Yukawa: Kwagaku (in Japanese), 12 (1942), 251, 282, 322.

(3) S. Tomonaga: Riken-Iho (in Japanese), 2 (1943), 545; Prog. Theor. Phys. 1 (1946), in press.

2. The g.t.f. in the quantum mechanics of particles.

We shall deal first in this section with the problem of quantum mechanics of particles in order to solve the one proposed in the preceding section.

Let us suppose a complete set of dynamical variables, a complete set of commuting observales in Dirac's notion;

$$q \equiv (q_1, q_2 \cdots\cdots, q_r).$$

As wellknown, the transformation function $\psi(q_T''/q_t')$ connects the two representations with the dynamical variables $q_t \equiv (q_{1t}, q_{2t}, \cdots, q_{\gamma t})$ at time t diagonal and the some dynamical variables $q_T \equiv (q_{1T}, q_{2T}, \cdots, q_{\gamma T})$ at time T $(T > t)$ diagonal respectively, and its square of the modulus $|\psi(q_T''|q_t')|^2$ is usually interpreted as the probability of an observation of the variables at the later time T giving the results q_T'' for a state for which an observation of the variables at the earlier time t is certain to give the results q_t'.

As Dirac already pointed out, we can alternatively consider $|\psi(q_T''|q_t')|^2$ as giving the relative a poriori probability of any state yielding the results q_t' and q_T'' when observations of the q's are made at time t and at time T (account being taken of the fact that the earlier observations will alter the state and affect the later observation). This latter interpretation is founded on the fundamental assumption of the quantum observations which states that the preparation of the state in which the q's have certainly the values q_t' is the same operation as the observation by which we obtain for the q_t's the values q_t'.

The transformation function $\psi(q_T''|q_t')$, considered as a function of variables q_T'', is a solution of Schrödinger's equation

$$i\hbar\frac{\partial}{\partial T}\psi(q_T''|q_t') = \sum_{q_T'''}(q_T''|H|q_T''')\psi(q_T'''|q_t') \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

with the initial condition $\psi(q_t''|q_t') = \delta_{q_t''q_t'}$ at time $T=t$, where H is the Hamiltonian operator of the system. The solution of the equation (1) may be written in the form

$$\psi(q_T''|q_t') = (q_T''|D(T-t)|q_t') \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

where $D(T-t)$ represents the transformation operator which transforms the state-vector ψ_t at time t to the state-vector ψ_T at time T:

$$\psi_T = D(T-t)\psi_t,$$

$$D(T-t) = \prod_t^T\left(1 - \frac{i}{\hbar}H\,dt\right) = \exp\left\{-\frac{i}{\hbar}H\cdot(T-t)\right\} \quad\cdots\cdots\cdots\cdots (3)$$

i.e. the transformation function $\psi(q_T''|q_t')$ is given by the matrix element of the mixed representation of the transformation operator $D(T-t)$, whose rows and columns refer to the q_T'''s and the q_t''s. As previously stated, we can interprete the square of the modulus of the transformation function $\psi(q_T''|q_t')=(q_T''|D(T-t)|q_t')$ as the relative a priori probability of any state yielding the results q_t' and q_T'' when observations of the q's are made at time t and at time T. This form is, however, refered to the special representation chosen and therefore it seems to be not in a suitable form for such an interpretation. Then we shall try to rewrite the transformation function $\psi(q_T''|q_t')$ in a form suitable for it.

Let us denote by Mq_t' and Mq_T'' the observation-opeators belonging to such observations that we obtain the results q_t' and q_T'' when observations of the q's are made at time t and at time T, the observation-operators Mq are Hermitian operators: $\widetilde{Mq'}=Mq'$, and satisfy $Mq'^2=Mq'$ and therefore have the propervalues 0 and 1. The operators Mq' mean that we represent a state-vector in the complete normalized orthogonal system belonging to the operators q and take out only the part depending on the proper values q' of the q's.

As the result of such a observation, any state changes to the state transformed by the transformation operator:

$$T(q_T''q_t')=Mq_T''D(T-t)Mq_t'. \quad \text{......} \quad (4)$$

Let us denote any representation of the transformation operator $T(q_T''q_t')$ by $(Q''|T(q_T''q_t')|Q')$, then we have

$$\sum_{Q''}\sum_{Q'}|(Q''|T(q_T''q_t')|Q')|^2=\sum_{Q''}\sum_{Q'}(Q'|\widetilde{T}(q_T''q_t')|Q'')(Q''|T(q_T''q_t')|Q')$$

$$=\text{Trace } \widetilde{T}(q_T''q_t')T(q_T''q_t')=|(q_T''|D(T-t)|q_t')|^2 \quad \text{......} \quad (5)$$

where we used the Hermitian property of Mq' and the relation $\widetilde{D}(T-t)=D(t-T)$ as is easily verified with the help of (3). The equation (5) holds independent on the representation chosen. We may consider $T(q_T''q_t')$ as a function of parameters q_t' and q_T'', we would, therefore, call $T(q_T''q_t')$ itself the generalized transformation function (to call the generalized transformation operator rather may become it).

Suppose we take between the time interval $t\rightarrow T$ a sequence of intermediate times $t_s\ (s=1, 2\ldots n)$:

$$t<t_1<t_2<\cdots<t_n<T,$$

and denote by

$$Mq_t', \ Mq_{t_1}', \ Mq_{t_2}', \ \ldots\ldots, \ Mq_{tn}', \ Mq_T'$$

the observation-operators belonging to the observations by which we obtain the results

$$q_t', \ q_{t_1}', \ q_{t_2}', \ \ldots\ldots, \ q_{tn}', \ q_T'$$

of

$$q_t, \ q_{t_1}, \ q_{t_2}, \ \ldots\ldots, \ q_{tn}, \ q_T$$

respectively. In this case the above defined g.t.f. can be given by

$$T(q_T' q_{tn}' \cdots q_{t_1}' q_t') = Mq_T' D(T-t_n) Mq_{tn}' \ldots Mq_{t_1}' D(t_1-t) Mq_t'. \qquad (6)$$

The relative a priori probability of the sets of the dynamical variables $q_T, q_{tn}, \ldots, q_{t_1}, q_t$ having the sets of the values $q_T', q_{tn}', \ldots, q_{t_1}', q_t'$ when observations of the q's are made at each time $t, t_1, \ldots, t_n, T$ will be

$$\begin{aligned} &\text{Trace } \widetilde{T}(q_T', q_{tn}' \cdots q_t', q_t') T(q_T' q_{tn}' \cdots q_{t_1}', q_t') \\ &\quad = | (q_T' | D(T-t_n) | q_{tn}') (q_{tn}' | D(t_n - t_{n-1}) | q_{tn-1}) \times \cdots \\ &\qquad \cdots \times (q_{t_2}' | D(t_2 - t_1) | q_{t_1}') (q_{t_1}' | D(t_1 - t) | q_t') |^2 . \ldots\ldots\ldots\ldots (7) \end{aligned}$$

Our foregoing g.t.f. can easily be generalized to the cases in which the q's are not a complete set of dynamical variables and also $q_t, q_{t_1}, \ldots, q_{tn}, q_T$ are the different physical quantities.(4) For an example, in such a case as a complete set of dynamical variables can be formed not only out of q, but also together with r, when we make observations only about the q's as above mentioned, we get the expression (6) of the g. t. f., but the relative a priori probability of the q's yielding the set of results $q_T', q_{tn}', \ldots, q_{t_1}', q_t'$ is given by the expression

$$\begin{aligned} &\text{Trace } \widetilde{T}(q_T' q_{tn}' \cdots q_{t_1}', q_t') T(q_T' q_{tn}' \cdots q_{t_1}' q_t') \\ &\quad = \sum_{r'_T r'_t} \Big| \sum_{r'_{t_1}, \cdots r'_{tn}} (q_T' r_T' | D(T-t_n) | q_{tn}' r_{tn}) (q_{tn}' r_{tn}' | D(t_n - t_{n-1}) | q'_{tn-1} r'_{tn-1}) \times \\ &\qquad \cdots \times (q_{t_2}' r_{t_2}' | D(t_2 - t_1) | q_{t_1}' r_{t_1}') (q_{t_1}' r_{t_1}' | D(t_1 - t) | q_t' r_t') \Big|^2 \ \ldots\ldots\ldots\ldots (8) \end{aligned}$$

Finally we shall consider the significance of the g. t. f. According to the relativistic sense of the quantum state the change of the state can take place only when we made an observation for the state. The above defined g. t. f. may be considered as connecting the two different states in the relativistic sense. Let us represent a state in the relativistic sense by a symbol Ψ and a state in the ordinary non-relativistic sense by a symbol ψ. The relation between Ψ and ψ may be given as follows. A relativistic Ψ is defined by the non-relativistic states which are represented by ψ_t at time t and by $\psi_T = D(T-t)\psi_t$ at any other time T. By the observations of the q_t''s and the q_T'''s for the state Ψ, the state of the system will change from Ψ to the state Ψ' which is completely determined by the corresponding non-relativestic states represented by

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(4) F. Bloch: Phys. Zs. d. Sowj., 5 (1934), 301.

$$\psi_t' = D(t-T)Mq_T''D(T-t)Mq_t'\psi_t \qquad \text{at time } t$$

and

$$\begin{aligned}\psi_T' &= D(T-t)\psi_t' = Mq_T''D(T-t)Mq_t'\psi_t \\ &= Mq_T''D(T-t)Mq_t'D(t-T)\psi_T. \qquad \text{at time } T.\end{aligned}$$

For the state Ψ' we obtain certainly the values q_t' of the q_t's at time t and the values q_T' of the q_T's at time T. The g. t. f. $T(q_T''q_t')$ should be completely defined as transforming a state Ψ to such a state Ψ' as above defined. When we, however, consider the quantum states in the non-relativistic sense, the explicit expression of the g.t.f. $T(q_T''q_t')$ is determined only but the arbitrary factor $D(\tau)$.

For $T(q_T''q_t')$ we may take $Mq_T''D(T-t)Mq_t'D(t-T)$, $D(t-T)Mq_T''D(T-t)Mq_t'$ or generally $D(\tau'-T)Mq_T''D(T-t)Mq_t'D(t-\tau)$ in place of $Mq_T''D(T-t)Mq_t'$. It is easily seen that taking any one of them for $T(q_T''q_t')$, Trace $\widetilde{T}(q_T''q_t')T(q_T''q_t')$ has the same value.

3. The g. t. f. in the quantum theory of fields.

The preceding section included all of the fundamental and necessary conceptions as well as the mathematical formalism of the g. t. f.'s. In the quantum theory of fields we need only a formal extension. In place of Heisenberg-Pauli's theory for the formulation of the quantum theory of fields we shall take here Tomonaga's theory (the super-many-time-theory) which seems to be the more general and fundamental one than former.

For simplicity, as represented in Fig. 1, we shall think in space-time a closed surface $AA'BB'$ which consists of a bottom of the space-like surface AA', a lid of the space-like surface BB' and a side-wall of the time-like surface $ABA'B'$, although the following argument may be applied in the same line also when the closed surface is of any shape. Let us take any surfaces C_A and C_B including the surfaces AA' and BB' respectively and a

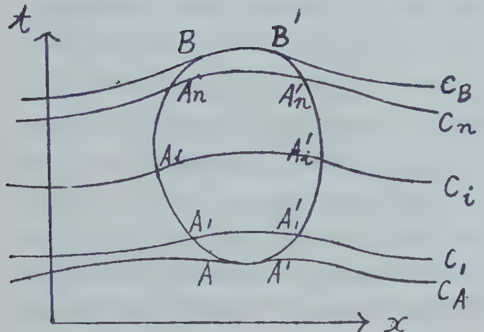

Fig. 1

sequence of any intermediate surfaces $C_i (i=1, 2, \ldots, n)$ between C_A and C_B, each extending to infinity in the space directions and lying entirely outside any light-cone having its vertex on the surface. We shall represent the closed lines (two-dimentional) being the intersections of the C_i's $(i=1, 2, \ldots, n)$ with the side-wall of the time-like surface $ABA'B'$ by $\widehat{A_iA_i'} (i=1, 2, \ldots, n)$. Let us denote by $Mv'(AA')$, $Mv'(\widehat{A_iA_i'})$ $(i=1, 2, \ldots, n)$ and $Mv'(BB')$ the observation-operators belonging to such observations that we obtain the specified results $v'(AA')$ of the field variables $v(AA') \equiv (v_1(AA'), v_2(AA'), \ldots v_\gamma(AA'))$ on the surface AA', $v'(\widehat{A_iA_i'})$ of the $v(\widehat{A_iA_i'})$'s on the closed lines $\widehat{A_iA_i'}$ $(i=1, 2, \ldots, n)$ and $v'(BB')$ of the $v(BB')$'s on the surface BB' respectively. The g.t.f. for the region bounded by the closed boundary surface C, which is interpretable as the amplitude of the relative a priori probability of the results $v'(C) \equiv v'(AA') + v'(ABA'B') + v'(BB')$ being obtained when observations are made of the field variables $v(C) \equiv v(AA') + v(ABA'B') + v(BB')$ on the boundary surface C, is now

$$T[v'(C)] = \lim_{n\to\infty} Mv'(BB')\, T[C_B;\ C_{n-1}]\, Mv'(\widehat{A_nA_n'}) \times \ldots\ldots$$

$$\ldots \times T[C_2;\ C_1]\, Mv'(\widehat{A_1A_1'})\, T[C_1;\ C_A]\, Mv'(AA') \qquad (9)$$

where

$$T[C_s;\ C_{s-1}] = \prod_{C_{s-1}}^{C_s} \left(1 - \frac{i}{\hbar} H' dw\right) \qquad (10)$$

is the transformotion operator in Tomonaga's theory, corresponding to $D(t_s - t_{s-1})$ in the preceding section transforming the state-vector $\Psi[C_{s-1}]$ on the surface C_{s-1} to the state-veetor $\Psi[C_s]$ on the surface C_s:

$$\Psi[C_s] = T[C_s;\ C_{s-1}]\, \Psi[C_{s-1}] \qquad (10')$$

H' represents the scalar density of the interaction energy, dw is a four-dimentional volume-element bounded by two space-like surfaces and $\prod_{C_{s-1}}^{C_s}$ means that we divide the region between the surface C_{s-1} and C_s into the very small sub-regions of dw's and make the repeated products of $\left(1 - \frac{i}{\hbar} H' dw\right)$'s progressively according to the time-order.

The relative a priori probability of the field variables $v(C)$ yielding the distribution of the values $v'(C)$ on the surface C may be represented by

$$\text{Trace}\ \widetilde{T}[v'(C)]\,T[v'(C)] \quad \cdots\cdots\cdots\cdots \quad (11)$$

Here, Trace means however only an abstract symbol and it is impossible to write the expression (11) in the form of the diagonal sum of the matrix in any suitable representation in the same manner of (5) or (7) in the ordinary quantum mechanics of particles. In order to express the equation (11) in such a manner, it is at least necessary to replace the continuously infinite freedoms of the field variables by the numerably infinite one by means of some methods.(3),(7)

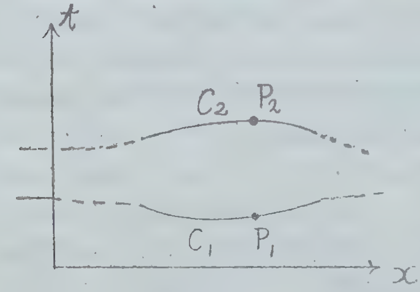

Fig. 2

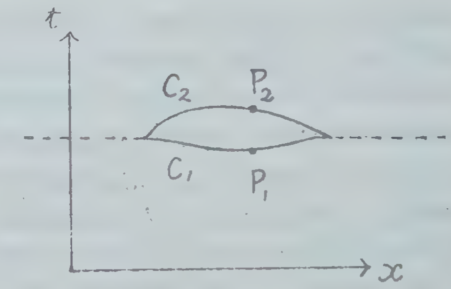

Fig. 3

For special cases, if we consider the distribution of the values of the field variables on the surfaces as represented in Fig. 2 or Fig. 3

$C=C_1+C_2$ (both of C_1 and C_2 are the space-like surfaces extending to infinity), the relative a priori probability of the $v(C)$'s having the values $v'(C)$ will be

$$\text{Trace}\ \widetilde{T}[v'(C)]\,T[v'(C)]=|[v'(P_2)\,|\,T[C_2,\ C_1]\,|\,v'(P_1)]|^2 \qquad (12)$$

where P_1 and P_2 are the parameters of the movable points on the parts of C_1 and C_2 where they are separated to each other. Thus the expression (12) agrees with the result being already obtained by Tomonaga.(3) Even in the case of two space-like surfaces C_1 and C_2 completely separated to each other in all space-time, as shown in Fig. 2, we may consider two surfaces as being connected at infinity and forming a closed surface C.

Finally we shall examine the composition law of the g. t. f. $T[v'(C)]$.

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(5) P. A. M. Dirac: Principles of Quantum Mechanics. Ist ed. 1930, 2nd. ed. 1934.

(6) W. Heisenberg, W. Pauli: Zs. f. Phys. 56 (1929) 1; 59 (1929), 168.

(7) P. Jordan, W. Pauli: Zs. f. Phys., 47 (1928), 151.

It will be seen that the composition law is not so general as Dirac⁽¹⁾ expected. We shall consider the case as shown in Fig. 1. The g. t. f.'s for the sub-regions bounded by the boundary surfaces $AA'A_i'A_i$ and $A_iA_i'B'B$ may be written

$$T[v'(AA'A_i'A_i)]=\lim Mv'(A_iA_i')\,T[C_i\,;\ C_{i-1}]\cdots\cdots$$
$$\cdots T[C_i\,;\ C_A]\,Mv'(AA')\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (13)$$

and similary

$$T[v'(A_iA_i'B'B)]=\lim Mv'(BB')\,T[C_B\,;\ C_n]\cdots\cdots$$
$$\cdots T[C_{i+1}\,;\ C_i]\,Mv'(A_iA_i').\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (13)'$$

Then the g. t. f. for the total region bounded by the surface C will be composed of the g.t.f.'s for the sub-regions which have a common boundary surface A_iA_i', namely

$$T[v'(C)]=\sum_{v'(A_iA_i')}T[v'(A_iA_i'BB)]\,T[v'(AA'A_iA_i')]\ \cdots\cdots\cdots\cdots\ (14)$$

where $\sum\limits_{v'(A_iA_i')}$ means that we sum up over all of the possible values of the field variables on the common boundary surface A_iA_i' of the two sub-regions. It is however noticeable that Dirac's composition law of the g. t. f. is satisfied, as above proved, only when the common boundary surface extends in the space-directions and lies entirely outside any light-cone having its vertex on the surface, and the total region is completely separated into two sub-regions by any space-like surface including the common surface and extending to infinity, for example C_i in Fig. 1. It is easily, according to our method of the construction of the g.t.f., seen that Dirac's composition law is not satisfied for arbitrary common surfaces, for an example in a case of the common surface including a part extending in the time-directions and lying inside any light-cone having its vertex on the surface, since the operators appearing in the g. t. f. are generally non-commutable at any two points in space-time either of which lies inside or on the light-cone from the other.

4. Conclusion

In our theory the observation operator appears in the forefront of the theory. This circumstance give us such an anticipation that in the future theory our practical observation may be expressed in a more explicit form. As a matter of fact it is exceptional that we obtain a complete knowledge of the state of the system by our practical observation and therefore we may say that our knowledge of the state is as usually of a statistical nature. This feature are expressed more directly in our theory than in the current theory. In fact we see that $\widetilde{T}T$ is a sort of statistical operators according to (8) and (11). But in so far as our theory is concerned it is not yet clear what amendments required in our theory in order to get rid of the difficulties existing in the theory of the elementary particles will be provided by the passage to the future theory, since the foregoing theory has been got only by a formal rewriting of the current theory and does not include any new ideas for amendments required for the passage to the future theory.

Our cordial thanks are due to Professors H. Yukawa and S. Sakata for their interest throughout in this work.

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On the Electromagnetic Properties of Nucleons, I.*

By Minoru KOBAYASI and Eizō KANAI.

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1. In the meson theory, the additional magnetic moments of protons and neutrons or the anomalous magnetic moments of nucleons have been attributed to the virtual emission of charged mesons by nucleons, and the numerical values of these quantities have been estimated by calculating the total energies of the nucleons placed in a weak homogeneous magnetic field. But, the result thus obtained by assuming the terms proportional to the field strength as those due to their magnetic moments diverges as in the case of the calculation of the self-energies of the elementary particles.(1) These diverging difficulties have been usually avoided by taking recourse to some "cutting off" prescription.

We should, however, not be content with such a provisory theory. Moreover, we have reason, which will be discussed in detail in the papers shortly published by us, to remove the diverging terms appearing in the calculation of self-energies completely. And, from such a standpoint, the terms which are proportional to the magnetic field strength and attributed to the energy due to the additional magnetic moments of nucleons should be removed simultaneously by the claim of gauge invariance. This circumstance is similar to the case of the theory of positrons where the diverging terms appearing in the calculation of the total energies of systems are removed in the relativistic and gauge invariant way.(2) The close analogy between the positron theory and the theory of the self-energy will be also discussed by us. Concluding from this circumstance, it seems doubtfull to us to account for the additional magnetic moments of nucleons in the usual way.

On the other hand, it is in general believed that the ordinary perturbation method of quantum theory applied to the calculation of the transition probability of any process gives the correct results in the first non-vanishing approximation, in spite of the lack of convergence of this method. This lack of convergence is also intimately connected with the diverging difficulty of self-energy and will disappear when the way out of the later is found.

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* The contents of this paper were read on Nov. 18, 1944, before the symposium of the National Research Council of Japan at the Tokyo Imperial University.

(1) Fröhlich, Heitler and Kommer, Proc. Roy. Soc. A. **166** (1938), 154. Yamasaki and Uma, Proc. Phys.-Math. Soc. Japan **24** (1942), 301. Ueno and Kuze, Ibid **24** (1942), 184.

(2) Dirac, Proc. Cambr. Phil. Soc. **30** (1934), 150. Heisenberg, Zeits. f. Phys. **90** (1934), 209. Weisskopf, D. Kagl. Danske Vidensk. Selskab, Math.-fys, Medd. XIV, 6. 1936.

Looking from this point of view, it seems interesting to us to calculate the transition probabilities of the processes which are due to the interaction of nucleon at rest with the electromagnetic waves and examine the electromagnetic properties of nucleons assuming they have the infinite rest masses. We can expect that if the so-called additional magnetic moments must inher in nucleons they give rise to any effect in these processes. Moreover, if there are other electromagnetic properties of nueleons, such as the electric dipol moment or electromagnetic higher moments, they will also contribute to the transition probabilities in the suitable approximations. Above all, it must be stressed that such a calculation will lead to the converging result in any case as long as it is restricted to the first non-vanishing approximation and give an unambiguous informations concerning electromagnetic properties of nucleons.

2. From the above points of view, we shall first pick up the process where the photon is scattered by the nucleon at rest, and calculate the transition probability of this process by assuming the rest mass of the nucleon to be infinite. Further, in this calculation the wave length of the incident photon will be assumed to be so long that the energy of the photon is negligible compared with the rest mass of a meson, and using the result in the limitting case where the energy of the photon tends to zero we will discuss the statical electromagnetic properties of the nucleon.

For this purpose, it is necessary to carry out the fourth order perturbation calculus straightforwardly for the transition processes caused by the interaction between the nucleon and the meson and that between the meson and the photon, and occationally the direct interaction among these three particles. For simplicity, let us assume the pseudo-scalar wave function for the meson field. Then, the interaction term in the Hamiltonian of the system is given by⁽³⁾

$$\bar{H}' = \int (H_{NM} + H_{ME} + H_{NME})\, dv \qquad \text{(1)}$$

where

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(3) Tanikawa and Yukawa, Proc. Phys.-Math. Soc. of Japan **23** (1941), 445.

$$H_{NM}=-\frac{f}{\varkappa^2}\left(\vec{P}\,\mathrm{grad}\,\tilde{\psi}+\tilde{\vec{P}}\,\mathrm{grad}\,\psi\right)\;\ldots\ldots\ldots\ldots\ldots(2a)$$

$$H_{ME}=\frac{ie}{4\pi\varkappa^2\hbar c}\left(\vec{A}\tilde{\psi}\,\mathrm{grad}\,\psi-\vec{A}\psi\,\mathrm{grad}\,\tilde{\psi}\right)$$

$$+\frac{1}{4\pi\varkappa^2}\left(\frac{e}{\hbar c}\right)^2\left(\vec{A}\tilde{\psi}\cdot\vec{A}\tilde{\psi}\right)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2b)$$

$$H_{NME}=\frac{ife}{\varkappa^2\hbar c}\left(\vec{A}\psi\tilde{\vec{P}}-\vec{A}\tilde{\psi}\vec{P}\right)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2c)$$

$$\vec{P}=\tilde{\varphi}\,\pi^*\vec{\sigma}\,\varphi\,,\quad \varkappa=\mu c/\hbar$$

represent the nucleon-meson, meson-photon and nucleon-meson-photon interaction respectively. In (2), ψ, $\tilde{\psi}$ and φ, $\tilde{\varphi}$ represent the wave functions and their complex conjugates of meson and nucleon respectively, and $\vec{A}$ represents the electromagnetic vector potential. Further, $\vec{\sigma}$ is the spin operator of nucleon and π and π^* are the operators which convert a proton into a neutron and vice versa and μ is the rest mass of a meson.

By expressing (2) in the quantised forms, and looking for all combinations which connect the initial state where the incident photon and the nucleon at rest are present with the final state where the photon is scattered in the different directions, we can find the intermediate states through which the scattering process takes place. In the expressions (2) for the interaction energy there are two kinds of terms, namely, the terms which involve the interaction constants e and f in the first power and those involving them in the second power. The former terms give rise to the scattering process in the fourth approximation, and the later terms in the second approximation. Moreover, the intercombinations of these two kinds of terms give rise to the scattering. We may classify the combinations of terms which lead to the scattering process into three kinds, namely (a) the combinations of four terms which have the constant e or f in the first power, (b) those involving the second term of (2b) as a factor, and (c) those which involve the term (2c). There are 4 ! combinations in the class (a), 6 combinations in the class (b), and 12 combinations in the class (c). The interaction matrix for the scattering process will he finally given by summing up these 44 terms. By making use of the approximation mentioned above and confining ourselves to the limiting case $\hbar\nu/\mu c^2\rightarrow 0$ (ν is the frequency of the incident photon), and integrating the terms all over the directions in which the mesons are virtually emitted, we get the following matrix elements corresponding to the above three kinds of combinations

$$\text{(a)}\quad -\frac{16}{3}\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot(\vec{e}_0\cdot\vec{e})\int dk\cdot k^6(k^2+x^2)^{-3}$$

$$\text{(b)}\quad 4\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot(\vec{e}_0\cdot\vec{e})\int dk\cdot k^4(k^2+x^2)^{-2}$$

$$\begin{aligned}\text{(c)}\quad &\frac{16}{3}\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot(\vec{e}_0\cdot\vec{e})\int dk\cdot k^4(k^2+x^2)^{-2}-4\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot(\vec{e}_0\cdot\vec{e})\int dk\cdot k^2(k^2+x^2)^{-1}\\ &+4i\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot([\vec{e}_0\cdot\vec{e}]\vec{\sigma})\int dk\cdot k^2(k^2+x^2)^{-3/2}\\ &-4i\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot([\vec{e}_0\cdot\vec{e}]\vec{\sigma})\int dk\cdot k^4(k^2+x^2)^{-5/2}\qquad(3)\end{aligned}$$

where $\vec{e}_0$ and $\vec{e}$ represent the directions of polarization of the incident and the scattered photons respectively, and k is the absolute value of the momentum of the virtually emitted meson devided by $\hbar$. In the integrations $\int dk$ over the absolute values of the momenta of mesons in the intermediate states, each expression of (3) diverges separately. But, these diverging terms cancel each other when they are added together, as it has been expected in the above discussions. By integrating (3) all over the momentum space of the virtually emitted mesons, we get, finally, the definite expression

$$\frac{4i}{3}\left(\frac{fe}{x}\right)^2\!/\hbar c\cdot([\vec{e}_0\cdot\vec{e}]\vec{\sigma})\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

for the matrix element of the scattering process of photons caused by the electromagnetic properties of the infinitely heavy nucleon. The scattering cross section will be given by multiplying the square of the absolute value of (4) by the density of the final state.

3. We shall now proceed to examine the result (4) closely and inquire whether it is possible or not to account for our result by attributing any electric or magnetic moment to the nucleon. It will be easily shown that if the nucleon has the magnetic moment $M\vec{\sigma}$ in its spin direction the matrix element corresponding to (4) is given by

$$-4\pi iM^2([\vec{b}_0\cdot\vec{b}]\cdot\vec{\sigma})\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $\vec{b}_0$ (or $\vec{b}$) denotes the direction which is perpendicular to $\vec{e}_0$ (or $\vec{e}$) and the direction of the incident (or scattered) photon, viz. the direction of the magnetic polarization of the corresponding photon. On the other hand, this matrix element is replaced by

$$-4\pi i N^2([\vec{e_0}\cdot\vec{e}]\cdot\vec{\sigma}) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

when the electric dipole moment $N\vec{\sigma}$ is attributed to the nucleon. Comparing these two expressions to (4), we can find that it is rather reasonable to treat the scattering process as it is caused by the electric moment, instead of the magnetic one, of the nucleon.

It seems, however, impossible to interpret this moment in the usual sense. Because, if we extract the square of the corresponding factor of (4) to get the expression for the electric moment itself, we shall find that it is given by the pure imaginary expression. In order to get an idea of the imaginary charactor of the electric moment, we have to remember Dirac's relativistic equation of electron, where the electric moment is given also by the imaginary expression involving a operator ρ_1 which connects the positive energy state with the negative one as a factor. This electric moment of a electron gives rise to any effect only when it is applied to such a second order process as the scattering of photons by the electron. In our case, the operator corresponding to ρ_1 is that which connects the proton state of nucleon with the state where a neutron and a positive meson are present or that connecting the neutron state with the corresponding state, according as the nucleon is a proton or a neutron. Denoting these operators symbolically by τ_1 and τ_2, the electric dipole moment of a proton or a neutron is given by

$$i\tau_1\,(\text{or }\tau_2)\frac{1}{\sqrt{3\pi\hbar c}}\,\frac{fe}{\varkappa}\vec{\sigma} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

as a non-diagonal matrix element which connects a proton or a nutron state with a imaginary dissociated state. It must, however, be borne in mind that such a dissociated state does not really appear in our process. Therefore, (7) should convey any meaning only when it is squared.

Neverthless, it must be stressed that, contrary to the case of Dirac's electron, the above obtained non-stationary electric moment of nucleon does not vanish even when the mass of a nucleon tends to infinity and plays a predominant rôle in any transition process caused by the interaction between the nucleon and the electromagnetic field. Further, it seems to be possible to examine the validity of the meson theory and estimate the magnitude of the interaction constants contained in the meson theory by comparing our result with the data which will been obtained by the suitably devised experiments.

Regarding the magnetic moment of the nucleon, we could not get any imformation in the above calculation. But, judging by the fact that, in the usual calculations cited above, the magnetic moment has been given by the expression which contains the interaction constants f in the second power, we can expect that it will play any rôle in the process whose order is higher than that of the simple scattering of photons by nucleons. We shall not, however, proceed with our plan in this way, because the calculations concerning such a complicated process may be very tedius. In the next paper, we shall treat this problem by the other method, namely, by the method of subtraction analogous to that of the theory of positrons. Also in this paper, the problem of the electric moment will be touched again from this standpoint, and the discussion of our present results will be made by comparing it with that obtained by the subtraction method.

In conclusion, it is our pleasant duty to express our thanks to Dr. Y. Nishina and Professor H. Yukawa for their kind interest and encouragement which have made this paper and following papers possible.

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On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields.*

By S. Tomonaga

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§1. The formalism of the ordinary quantum theory of wave fields.

Recently Yukawa(1) has made a comprehensive consideration about the basis of the quantum theory of wave fields. In his article he has pointed out the fact that the existing formalism of the quantum field theory is not yet perfectly relativistic.

Let $v(xyz)$ be the quantity specifying the field, and $\lambda(xyz)$ denote its canonical conjugate. Then the quantum theory requires the commutation relations of the form:

$$\begin{cases}[v(xyzt),\ v(x'y'z't)]=[\lambda(xyzt),\ \lambda(x'y'z't)]=0\\ [v(xyzt),\ \lambda(x'y'z't)]=i\hbar\delta(x-x')\delta(y-y')\delta(z-z'),\end{cases}\tag*{(1)**}$$

but these have quite non-relativistic forms.

The equations (1) give namely the commutation relations between the quantities at different points (xyz) and $(x'y'z')$ at the same instant of time t. The concept "same instant of time at different points" has, however, a definite meaning only one specifies some definite Lorentz frame of reference. Thus this is not a relativistically invariant concept.

Further, the Schrödinger equation for the ψ-vector representing the state of the system has the form;

$$\left(\bar{H}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi=0,\tag{2}$$

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[*] Translated from the paper, Bull. I. P. C. R. (Riken-iho), **22** (1943), 545, appeared originally in Japanese.

[**] $[A,\ B]=AB-BA$. We assume that the field obeys the Bose statistics. Our considerations apply also to the case of Fermi statistics.

where $\bar{H}$ is the operator representing the total energy of the field which is given by the space integral of a function of v and λ. As we adopt here the Schrödinger picture, v and λ are operators independent of time. The vector representing the state is in this picture a function of the time, and its dependence on t is determined by (2).

Also the differential equation (2) is no less non-relativistic. In this equation the time variable t plays a role quite distinguished from the space coordinates x, y and z. This situation is closely connected with the fact that the notion of probability amplitude does not fit with the relativity theory.

As is well known, the vector ψ has, as the probability amplitude, the following physical meaning: Suppose the representation which makes the field quantity $v(xyz)$ diagonal. Let $\psi[v'(xyz)]$ denote the representative of ψ in this representation.* Then the representative $\psi[v'(xyz)]$ is called probability amplitude, and its absolute square

$$W[v'(xyz)] = |\psi[v'(xyz)]|^2 \qquad (3)$$

gives the relative probability of $v(xyz)$ having the specified functional form $v'(xyz)$ at the instant of time t. In other words: Suppose a plane** which is parallel to the xyz-plane and intersepts the time axis at t. Then the probability that the field has the specified functional form $v'(xyz)$ on this plane is given by (3).

As one sees, a plane parallel to the xyz-plane plays here a significant role. But such a plane is only defined by referring to a certain frame of reference. Thus the probability amplitude is not a relativistically invariant concept in the space-time world.

§ 2. Four-dimensinal form of the commutation relations.

As stated above, the laws of the quantum theory of wave fields are usually expressed as mathematical relations between quantities having their meanings only in some specified Lorentz frame of reference. But since it is proved that the whole contents of the theory are of cource relativistically invariant, it must be certainly possible to build up the theory on the basis of concepts having relativistic space-time meanings. Thus, in his consideration, Yukawa has required with Dirac⁽²⁾ to generalize the notion of probability amplitude so that it fits with the relativity theory. We shall now show below that the generalization of the theory on these lines is in fact possible to the relativistically necessary and sufficient extent. Our results are, however, not so general as expected by Dirac and by Yukawa, but are already sufficiently general in so far as it is required by the relativity theory.

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* We use the square blackets to indicate a functional. Thus $\psi[v'(xyz)]$ means that ψ is a functional of the variable function $v'(xyz)$. When we use ordinary blackets ( ), as $\psi(v'(xyz))$, we consider ψ as an ordinary function of the function $v'(xyz)$. For example: the energy density is written as $H(v(xyz), \lambda(xyz))$ and this is also a function of x, y and z, whereas the total energy $H = \int H(v(xyz), \lambda(xyz))dv$ is a functional of $v(xyz)$ and $\lambda(xyz)$ and is written as $\bar{H}[v(xyz), \lambda(xyz)]$.

** We call a three-dimensional minifold in the four-dimensional space-time world simply "surface".

Let us suppose for simplicity that there are only two fields interacting with each other. The case of more number of fields can also be treated in the same way. Let v_1 and v_2 denote the quantities specifying the fields. The canonically conjugate quantities be λ_1 and λ_2 respectively. Then between these quantities the commutation relations

$$\begin{cases} [v_r(xyzt),\ v_s(x'y'z't)]=0 \\ [\lambda_r(xyzt),\ \lambda_s(x'y'z't)=0 & r,\ s=1,\ 2 \qquad (4) \\ [v_r(xyzt),\ \lambda_s(x'y'z't)]=i\hbar\delta(x-x')\delta(y-y')\delta(z-z')\delta_{rs} \end{cases}$$

must hold. The ψ-vector satisfies the Schrödinger equation

$$\left(\overline{H}_1+\overline{H}_2+\overline{H}_{12}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi=0. \qquad (5)$$

In this equation $\overline{H}_1$ and $\overline{H}_2$ mean respectively the energy of the first and the second field. $\overline{H}_1$ is given by the space integral of a function of v_1 and λ_1, $\overline{H}_2$ by the space integral of a function of v_2 and λ_2. Further, $\overline{H}_{12}$ is the interaction energy of the fields and is given by the space integral of a function of both v_1, λ_1 and v_2, λ_2. We assume (i) that the integrand of $\overline{H}_{12}$, i. e. the interaction-energy density, is a scalar quantity, and (ii) that the energy densities at two different points (but at the same instant of time) commute with eath other. In general, these two facts follow from the single assumption: the interaction term in the Lagrangean does not contain the time derivatives of v_1 and v_2.

If this energy density is denoted by H_{12}, then we have

$$\overline{H}_{12}=\int H_{12}dx\,dy\,dz. \tag{6}$$

As we adopt here the Schrödinger picture, the quantities v and λ in H_1, H_2 and H_{12} are all operators independent of time.

Thus far we have merely summarized the well known facts. Now, as the first stage of making the theory relativistic, we suppose the unitary operator

$$U=\exp\left\{\frac{i}{\hbar}(\overline{H}_1+\overline{H}_2)t\right\} \tag{7}$$

and introduce the following unitary transformations ot v and λ, and the corresponding transformation of ψ:

$$\begin{cases} V_r=Uv_rU^{-1}, \quad \Lambda_r=U\lambda_rU^{-1} \\ \qquad \Psi=U\psi. \end{cases} \qquad r=1,\ 2 \tag{8}$$

As stated above, v and λ in (5) are quantities independent of time. But V and Λ obtained from them by means of (8) contain t through U. Thus they depend on t by

$$\begin{cases} i\hbar\dot{V}_r=V_r\overline{H}_r-\overline{H}_rV_r \\ i\hbar\dot{\Lambda}_r=\Lambda_r\overline{H}_r-\overline{H}_r\Lambda_r. \end{cases} \qquad r=1,\ 2 \tag{9}$$

These equations must necessarily have covariant forms against Lorentz transformations, because they are just the field equations for the fields when they are left alone without interacting with eath other.

Now, the solutions of these "vacuum equations", the equations which the fields must satisfy when they are left alone, together with the commutaion relations (4), give rise to the relations of the following forms:

$$\begin{cases} [V_r(xyzt),\ V_s(x'y'z't')]=A_{rs}(x-x\ ,\ y-y',\ z-z'\ \ t-t') \\ [\Lambda_r(xyzt),\ \Lambda_s(x'y'z't')]=B_{rs}(x-x',\ y-y',\ z-z',\ t-t') \\ [V_r(xyzt),\ \Lambda_s(x'y'z't')]=C_{rs}(x-x',\ y-y',\ z-z',\ t-t') \end{cases} \tag{10}$$

where A_{rs}, B_{rs} and C_{rs} are functions which are combinations of the so-called four-dimensional δ-functions and their derivatives.(3) One denotes usually these four dimensional δ-functions by $D_r(xyzt)$, $r=1,\ 2$. They are defined by

$$D_r(xyzt)=\frac{1}{16\pi^3}\iiint\left\{\frac{e^{i(k_x x+k_y y+k_z z+ck_r t)}}{ik_r}-\frac{e^{i(k_x x+k_y y+k_z z-ck_r t)}}{ik_r}\right\}dk_x dk_y dk_z \qquad (11)$$

with

$$k_r=\sqrt{k_x^2+k_y^2+k_z+k_r^2}\,, \qquad (12)$$

κ_r being the constant characteristic to the field r. It can be easily proved that these functions are relativistically invariant.*

Since (10) gives, in contrast with (4), the commutation relations between the fields at two different world points $(xyzt)$ and $(x'y'z't')$, it contains no more the notion of same instant of time. Therefore, (10) is sufficiently relativistic presupposing no special frame of reference. We call (10) fourdimensional form of the commutation relations.

One property of $D(xyzt)$ will be mentioned here: When the world point $(xyzt)$ lies outside the light cone whose vertex is at the origin, then $D(xyzt)$ vanishes identically:

$$D(xyzt)=0 \quad \text{for} \quad x^2+y^2+z^2-c^2t^2>0. \qquad (13)$$

It follows directly from (13) that, if the world point $(x'y'z't')$ lies outside the light cone whose vertex is at the world point $(xyzt)$, the right-hand sides of (10) always vanish. In words: Suppose two world points P and P'. When these points lie outside each other's light cones, the field quantities at P and field quantities at P' commute with eath other.

§ 3. Generalization of the Schrödinger equation.

Next we observe the vector Ψ obtained from ψ by means of the unitary transformation U. We see from (5), (7) and (8) that this Ψ, considered as a function of t, satisfies

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* Suppose that a surface in the $k_x\,k_y\,k_z\,k$-space is defined by means of the equation $k^2=k^2_x+k^2_y+k^2_z+\kappa^2$. Then this surface has the invariant meaning in this space, since $k^2_x+k^2_y+k^2_z-k^2$ is invariant against Lorentz transformations. The area of the surface element of this surface is given by $dS=\sqrt{\left(\frac{\partial k}{\partial k_x}\right)^2+\left(\frac{\partial k}{\partial k_y}\right)^2+\left(\frac{\partial k}{\partial k_z}\right)^2-1}\;dk_x\,dk_y\,dk_z$ $=,\kappa\frac{dk_x\,dk_y\,dk_z}{k}$. Now, since dS has the invariant meaning, we can thus conclude that $\frac{dk_x\,dk_y\,dk_z}{k}$ is an invariant, and this results that the function defined by (11) is invariant.

$$\left\{\iint H_{12}(V_1(xyzt),\ A_1(xyzt),\ V_2(xyzt),\ A_2(xyzt))\,dx\,dy\,dz + \frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\Psi=0. \qquad (14)$$

One sees that t plays also here a role distinguished from x, y and z: also here a plane parallel to the xyz-plane has a special significance. So we must in some way remove this unsatisfactory feature of the theory.

This improvement can be attained in the way similar to that in which Dirac(4) has built up the so-called many-time formalism of the quantum mechanics. We will now recall this theory.

The Schrödinger equation for the system containing N charged particles interacting with the electromagnetic field is given by

$$\left\{\overline{H}_{el}+\sum_{n=1}^{N}H_n(q_n,\ p_n,\ \mathfrak{a}(q_n))+\frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\psi=0. \qquad (15)$$

Here $\overline{H}_{el}$ means the energy of the electromagnetic field, H_n the energy of the n-th particle. H_n contains, besides the kinetic energy of the n-th particle, the interaction energy between this particle and the field through $\mathfrak{a}(\gamma_r)$, q_n being the coordinates of the particle and $\mathfrak{a}$ the potential of the field. p_n in (15) means as usual the momentum of the n-th particle.

We consider now the unitary operator

$$u=\exp\left\{\frac{i}{\hbar}\overline{H}_{el}t\right\} \qquad (16)$$

and introduce the unitary transformation of $\mathfrak{a}$:

$$\mathfrak{A}=u\mathfrak{a}u^{-1} \qquad (17)$$

and the corresponding transformation of ψ:

$$\Phi=u\psi. \qquad (18)$$

Then we see that Φ satisfies the equation

$$\left\{\sum_n H_n(q_n,\ p_n,\ \mathfrak{A}(q_n,\ t))+\frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\Phi=0. \qquad (19)$$

In contrast with $\mathfrak{a}$, which was independent of times (Schrödinger picture), $\mathfrak{A}$

contains t through u. To emphsize this, we have written t explicitely as argument of $\mathfrak{A}$. We can prove that $\mathfrak{A}$ satisfles the maxwell equations in vacuum (accurately speaking, we need special considerattons for the equation div $\mathfrak{E}=0$).

The equation (19) is the starting point of the many-time theory. In this theory one introduces then the function $\Phi(q_1t_1, q_2t_2, \dots, q_N, t_N)$ containing so many time variables $t_1, t_2, \dots\dots t_N$ as the number of the particles in place of the function $\Phi(q_1, q_2, \dots, q_N, t)$ containing only one time variable,* and suppose that this $\Phi(q_1t_1, q_2t_2, \dots, q_Nt_N)$ satisfies simultaneously the following N equations;

$$\left\{H_n(q_n, p_n, \mathfrak{A}(q_n, t_n)) + \frac{\hbar}{i}\frac{\partial}{\partial t_n}\right\}\Phi(q_1t_1, q_2t_2, \dots, q_Nt_N)=0$$

$$n=1, 2, \dots, N. \tag{20}$$

This $\Phi(t_1, t_2, \dots, t_N)$, which is a fundamental quantity in the many-time theory, is related to the ordinary probability amplitude $\Phi(t)$ by

$$\Phi(t)=\Phi(t, t, \dots, t). \tag{21}$$

Now, the simultaneous equations (20) can be solved when and only when the N^2 conditions

$$(H_nH_{n'}-H_{n'}H_n)\Phi(q_1t_1, q_2t_2, \dots, q_Nt_N)=0 \tag{22}$$

are satisfied for all pairs of n and n'. If the world point (q_nt_n) lies outside the light cone whose vertex is at the point $(q_{n'}t_{n'})$, we can prove $H_nH_{n'}-H_{n'}H_n=0$. As the result, the function satisfying (20) can exist in the region where

$$(q_n-q_{n'})^2-c^2(t_n-t_{n'})^2\geqq 0 \tag{23}$$

is satisfied simultaneously for all values of n and n'.

According to Bloch$^{(5)}$ we can give $\Phi(q_1t_1\ q_2t_2, \dots, q_Nt_N)$ a physical meaning when its arguments lie in the region given by (23). Namely

$$W(q_1t_1, q_2t_2, \dots\ q_Nt_N)=|\Phi(q_1t_1, q_2t_2, \dots, q_Nt_N)|^2 \tag{24}$$

gives the relative probability that one finds the value q_1 in the measurement of the position of the first particle at the instant of time t_1, the value q_2 in the measurement of the position of the second particle at the instant of time t_2, ... and the value q_N in the measurement of the position of the N-th particale at the instant of time t_N.

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* Here we suppose the representation which makes the coordinates $q_1, q_2, \dots, q_N$ diagonal. Thus the vector Φ is represented by a function of these coordinates.

This is the outline of the many-time formalism of the quantum mechanics.

We will now return to our main subject. If we compare our equation (14) with the equation (19) of the many-time theory, we notice a marked similarity between these two equations. In (19) stands the suffix n, which designates the particle, while in (14) stand the variables x, y and z, which designate the position in space. Further, Φ is a function of the N independent variables q_1, q_2, ..., q_N, q_n giving the position of the n-th particle, while Ψ is a functional of the infinitely many "independent variables" $v_1(xyz)$ and $v_2(xyz)$, $v_1(xyz)$ and $v_2(xyz)$ giving the fields at the position (xyz). Corresponding to the sum $\sum\limits_n H_n$ in (19) the integral $\int H_{12}dxdydz$ stands in (14). In this way, to the suffix n in (19) which takes the values 1, 2, 3, ..., N correspond the variables x, y and z whice take continuously all values from $-\infty$ to $+\infty$

Such a similarity suggests us to introduce infinitely many time variables t_{xyz}, which we may call local time * each for one position (xyz) in the space as we have introduced N time variables, particle times, t_1, t_2, ... t_N, each for one particle. The only difference consist in that we use in our case infinitely many time variables whereas we have used N time variables in the ordinary many-time theory

Corresponding to the transition from the use of the function with one time variable to the use of the function of N time variables, we must now consider the transition from the use of $\Psi(t)$ to the use of a functional $\Psi[t_{xyz}]$ of infinitely many time variables t_{xyz}.

We regard now t_{xyz} as a function of (xyz) and consider its variation ε_{xyz} which differs from zero only in a small domain V_0 in the neighbourhood of the point $(x_0y_0z_0)$. We will define the partical differential coefficient of the functional $\Psi[t_{xyz}]$ with respect to the variable $t_{x_0y_0z_0}$ in the following manner:

$$\frac{\delta\Psi}{\delta t_{x_0y_0z_0}} = \lim_{\substack{\varepsilon\to 0\\ V_0\to 0}} \frac{\Psi[t_{xyz}+\varepsilon_{xyz}]-\Psi[t_{xyt}]}{\iiint \varepsilon_{xyx}\, dx\, dy\, dz} \tag{25}$$

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* The notion of local time of this kind has been occasionary introduced by Stueckerberg.⁽⁶⁾

We then generalize (14), and regard

$$\left\{H_{12}(x,\ y,\ z,\ t)+\frac{\hbar}{i}\frac{\delta}{\delta t_{xyz}}\right\}\Psi=0 \tag{26}$$

the infinitely many simultaneous equations corresponding to the N equations (20), as the fundamental equations of our theory. In (26) we have written, for simplicity. $H_{12}(x,\ y,\ z,\ t)$ in place of $H_{12}(V_1(xyz,\ t),\ V_2(xyz,\ t),\ \ldots\ldots)$. In general, when we have a function $F(V,\ \Lambda)$ of V and Λ, we will write simply $F(x,\ y,\ z,\ t)$ for $F(V(xyz,\ t_{xyz}),\ \Lambda(xyz,\ t_{xyz}))$, or still simpler $F(P)$ P denoting the world point with the coordinates $(xyz,\ t_{xyz})$. Thus $F(P')$ means $F(x',y',z',t')$ or, more precisely, $F(V(x'y'z',\ t_{x'y'z'}),\ (x'y'z',\ {}_{x'y'z'}))$.

We will now adopt the equation (26) as the basis of our theory. For $V_1(P)$, $V_2(P)$, $\Lambda_1(F)$ and $\Lambda_2(P)$ in H_{12} the commutation relations (10) hold, where $D(xyzt)$ has the property (13). As the consequence, we have

$$H_{12}(P)H_{12}(P')-H_{12}(P')H_{12}(P)=0 \tag{27}$$

when the point P lies a finite distance apart from P' and outside the light cone whose vertex is at P. Further, from our assumption (ii) the relation (27) holds also when P and P' are two adjacent points approaching in a space-like direction. Thus our system of equations (26) is integrable when the surface defined by the equations $t=t_{xyz}$, considering t_{xyz} as a function of x, y and z, is space-like.

In this way, a functional of the variable surface in the space-time world is determined by the functional partial differential equations (26). Corresponding to the relation (21) in case of many-time theory, $\Psi[t_{xyz}]$ reduces to the ordinary $\Psi(t)$ when the surface reduces to a plane parallel to the xyz-plane.

The dependent variable surface $t=t_{xyz}$ can be of any (space-like) form in the space-time world, and we need not presuppose any Lorenz frame of reference to define such a surface. Therefore, this $\Psi[t_{xyz}]$ is a relativistically invariant concept. The restriction that the surface must be space-like makes no harm since the property that a surface is space-like or time-like does not depend on a special choise of the reference system. It is not necessary, from the stand-point of the relativity theory, to admit also time-like surfaces for the variable surface, what was required by Dirac and by Yukawa. Thus we consider that $\Psi[t_{xyz}]$ introduced above is already the sufficient generalization of the ordinary ψ-vector, and assume that the quantum-theoretical state* of the fields is represented by this functional vector.

Let C denote the surface defined by the equation $t=t_{xyz}$. Then Ψ is a functional of the surface C. We write this as $\Psi[C]$. On C we take a point P, whose coordinates are (xyz, t_{xyz}), and suppose a surface C' which overlap C except in a small domain about P. We denote the volume of the small world lying between C and C' with $d\omega_P$. Then we may write (25) also in the form:

$$\frac{\delta\Psi[C]}{\delta C_P}=\lim_{C'\to C}\frac{\Psi[C']-\Psi[C]}{d\omega_P}. \tag{28}$$

Then (26) can be written in the form:

$$\left\{H_{12}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0. \tag{29}$$

This equation (29) has now a perfect space-time form. In the first place, H_{12} is a scalar according to our assumption (i); in the second place, the commutation relations between $V(P)$ and $A(P)$ contained in H_{12} has the four-dimensional forms as (10), and finally the differentiation $\frac{\delta}{\delta C_P}$ is defined by (28) quite independently of any frame of reference.

A direct conclusion obrained from (29) is that $\Psi[C']$ is obtained from $\Psi[C]$ by the following infinitesimal transformation:

$$\Psi[C']=\left\{1-\frac{i}{\hbar}H_{12}(P)d\omega_P\right\}\Psi[C]. \tag{30}$$

When there exist in the space-time world two surfaces C_1 and C_2 a finite distance apart, we need only to repeat the infinitesimal transformations in order to obtain $\Psi[C_2]$ from $\Psi[C_1]$. Thus

$$\Psi[C_2]=\prod_{C_1}^{C_2}\left\{1-\frac{i}{\hbar}H_{12}(P)d\omega_P\right\}\Psi[C_1] \tag{31}$$

The meaning of this equation is as follows: We devide the world region lying between C_1 and C_2 in small elemdnts $d\omega_P$ (it is necessary that each world element is surrounded by two space-like surfaces). We consider for each world element the infinitesimal transformation $1-\frac{i}{\hbar}H_{12}(P)d\omega_P$. Then we take the product of these transformations, the order of the factor being taken from C_1 to C_2. This product transforms then $\Psi[C_1]$ into $\Psi[C_2]$.

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* The word state is here used in the relativistic space-time meaning. Cf. Dirac's book (second eddition) § 6.

The surfaces C_1 and C_2 must be here both space-like, but otherwise they may have any form and any configuration. Thus C_2 does not necessarily lie afterward against C_1; C_1 and C_2 may even cross with each other.

The relation of the form (31) has been already introduced by Heisenberg.(7) It can be regarded as the integral form of our generalized Schrödinger equation (29)

§ 4. Generalized probability amplitude.

We must now find the physical meaning of the functional $\Psi[C]$. As regards this we can make a similar consideration as Bloch has done for the case of ordinary many-time theory. Besides the fact that in our case there appear infinitely many time variables, one point differs from Bloch's case that in (16) the unitary operator u is commutable with the coordinates q_1, q_2, ... q_N, our U is not commutable with the field quantities $v_1(xyz)$ and $v_2(xyz)$. Noting this difference and treating the continuum infinity as the limit of an ennumerable infinity by some artifice, for instance, by the procedure of Heisenberg and Pauli,(8) Bloch's consideration can be applied also here almost without any alteration. We shall give here only the results.

Let us suppose that the fields are in the state represented by a vector $\Psi[C]$. We suppose that we make measurments of a function $f(v_1, v_2, \lambda_1, \lambda_2)$ at every point on a surface C_1 in the space-time world. Let P_1 denote the variable point on C_1, then, if $f(P_1)$ at any two "values" of P_1 commute with each other, the measurement of f at each of these two points do not interfere with each other. Our first conclusion says that in this case the expectation value of $f(P_1)$ is given by

$$\overline{f(P_1)}=((\Psi[C_1], f(P_1)\Psi[C_1])) \tag{32}$$

where $f(P_1)$ means $f(V_1(P_1), \ldots\ldots)$ according to our convention on page 35, and the symbol $((A, B))$ with double blackets is the scalar product of two vectors A and B. It is impossible in case of continuously many degree of freedom to represent this scalar product by an integral of the product of two functions. For this purpose we must replace the continuum infinity by an at least ennumerable infinity.

More generally, we suppose a functional $F[f(P_1)]$ of the independent variable function $f(P_1)$, regarding $f(P_1)$ as a function of P_1. Then the expectation value of this F is given by

$$\overline{F[f(P_1)]}=((\Psi[C_1],\ F[f(P_1)]\,\Psi[C_1])). \tag{33}$$

A physically interesting F is the projective operator $M[v_1'(P_1), v_2'(P_1), V_1(P_1), V_2(P_1)]$ belonging to the "eigen-value" $v_1'(P_1)$, $v_2'(P_1)$ of $V_1(P_1)$, $V_2(P_1)$. Then its expectation value

$$\begin{aligned}&\overline{M[v_1'(P_1), v_2'(P_1);\ V_1(P_1), V_2(P_1)]}\\&\qquad=((\Psi[C_1],\ M[v_1'(P_1), v_2'(P_1)\ ;\ V_1(P_1), V_2(P_1)]\Psi[C_1]))\end{aligned} \tag{34}$$

gives the probability that the field 1 and the field 2 have respectively the functional form $v_1'(P_1)$ and $v_2'(P_1)$ on the surface C_1. As C_1 is assumed to be space-like, the measurement of the functional M is possible (the measurements of $V_1(P_1)$ and $V_2(P_1)$ at all points on C_1 mean just the measurement of M).

Thus far we have made no mention of the representation of $\Psi[C]$. We use now the special representation in which $V_1(P_1)$ at all points on C_1 are simultaneously diagonal. It is always possible to make all $V_1(P_1)$ and $V_2(P_1)$ diagonal when the surface C_1 is space-like. In this representation $\Psi[C_1]$ is represented by a functional $\Psi[v_1'(P_1), v_2'(P_1)\ ;\ C_1]$ of the eigen-values $v_1'(P_1)$ and $v_2'(P_1)$ of $V_1(P_1)$ and $V_2(P_1)$. The projection operator M has in this representation such diagonal form that (34) is simplified as follows

$$\begin{aligned}W[v_1'(P_1), v_2'(P_1)]&=\overline{M[v_1'(P_1), v_2'(P_1);\ V_1(P_1), V_2(P_1)]}\\&=|\,\Psi[v_1'(P_1),\ v_2'(P_1);\ C_1]\,|^2.\end{aligned} \tag{35}$$

In this sence we can call $\Psi[v_1'(P_1), v_2'(P_1);\ C_1]$ "generalized probability amplitude".

§ 5. Generalized transformation functional.

We have stated adove that between $\Psi[C_1]$ and $\Psi[C_2]$ the relation (31) holds, where C_1 and C_2 are two spece-like surfaces in the space-time world. We see thus that the transformation operator

$$T[C_2;\ C_1]=\prod_{C_1}^{C_2}\left(1-\frac{i}{\hbar}H_{12}d\omega\right) \tag{36}$$

plays an important role. It is evident that also this operator has a space-time meaning.

Similarly as the special representative of the ψ-vector, the probability amplitude, has a distinct physical meaning, there is a special representation in which the representative of the transformation operator $T[C_2;\ C_1]$ has a distinct physical meaning.

We introduce namely the mixed representative of $T[C_2;\ C_1]$ whose rows refer to the representation in which $V_1(P_1)$ and $V_2(P_1)$ at all points on C_1 become diagonal and whose column refer to the representation in which $V_1(P_2)$ and $V_2(P_2)$ at all potnts on C_2 become diagonal. We denote this representation by

$$[v_1''(P_2),\ v_2''(P_2) \mid I[C_2;\ C_1] \mid v_1'(P_1),\ v_2'(P_1)], \quad (37)^*$$

or simpler:

$$[v_1''(P_2),\ v_2''(P_2) \mid v_1'(P_1),\ v_2'(P_1)]. \quad (38)^*$$

If we note here the relation (35), we see that we can give the matrix elements of this representation the following meaning: One measures the field quantities V_1 and V_2 at all points on C_2 when the fields are prepared in such a way that they have certainly the values $v_1'(P_1)$ and $v_2'(P_1)$ at all points on C_1. Then

$$\begin{aligned}&W[v_1''(P_2),\ v_2''(P_2)\ ;\ v_1'(P_1),\ v_2'(P_1)]\\&\quad=|\,[v_1''(P_2),\ v_2''(P_2) \mid v_1'(P_1),\ v_2'(P_1)]\,|^2\end{aligned} \tag{39}$$

gives the probability that one obiains the result $v_1''(P_2)$ and $v_2''(P_2)$ in this measurement. In this proposition we have assumed that C_2 lies afterward against C_1.

From this physical interpretation we may regard the matrix element (37), or (38), considered as a functional of $v_1''(P_2)$, $v_2''(P_2)$ and $v_1'(P_1)$, $v_2'(P_1)$, as the generalization of the ordinary transformation function $(q_{t_2}'' \mid q_{t_1}')$.

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* As the matrix elements are functionals of $v(P)$, we use here the square blackets.

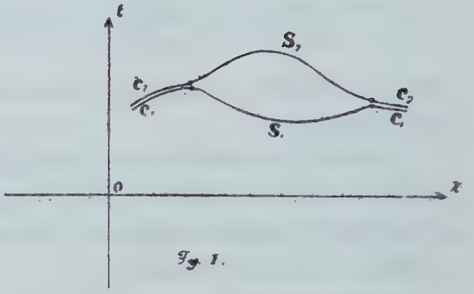

Fig. 1.

As a special case it may happen that C_2 lies apart from C_1 only in a portion S_2 and a portion S_1 of G_2 and C_1 respectively, the other parts of C_1 and C_2 overlapping with each other (see Fig. 1).

In this case the matrix elements of $T[C_2; C_1]$ depend only on the values of the fields on the portions S_1 and S_2 of the surfaces C_1 and C_2. In this case we need for calculating $T[C_2; C_1]$ to take the product in (36) only in the closed domain surrounded by S_1 and S_2, thus

$$T[S_2;\ S_1]=\prod_{S_1}^{S_2}\left(1-\frac{i}{\hbar}H_{12}d\omega\right). \tag{40}$$

The matrix elements of the mixed representation of this T is a functional of $v_1'(p_1)$, $v_2'(p_1)$ and $v_1''(p_2)$, $v_2''(p_2)$ where p_1 denots the moving point on the portion S_1, and p_2 the moving point on the portion S_2. This matrix is independent on the field quantities on the other portions of the surfaces C_1 and C_2.

The matrix element of $T[S_2; S_1]$ regarded as a functional of $v_1'(p_1)$, $v_2'(p_1)$ and $v_1''(p_2)$, $v_2''(p_2)$ has the properties of g. t. f. (generalized transformation functional) of Dirac. But in defining our g.t.f. we had to restrict the surfaces S_1 and S_2 to be space-like, while Dirac has required his g. t. f. to be defined also referring to the time-like surfaces. As mentioned above, however, such a generalization as required by Dirac is superflous so far as the relativity theory concerns.

It is to be noted that for the physical interpretation of $[v_1''(P_2),\ v_2''(P_2)|\ v_1'(P_1),\ v_2'(P_1)]$ it is not necessary to assume C_2 to lie afterward against C_1. Also when the inverse is the case, we can as well give the physical meaning for W of (39): One measures the field quantities V_1 and V at all points on C_2 when the fields are prepared in such a way that they would have certainly the values $v_1'(P_1)$ and $v_2'(P_1)$ at all points on C_1 if the fields were left alone until C_1 without being measured before on C_2. Then W gives the probability that one finds the results $v_1''(P_2)$ and $v_2''(P_2)$ in this measurement on C_2.

§ 6. Concluding remark.

We have thus shown that the quantum tneory of wave fields can be really brought into a form which reveals directly the invariance of the theory against Lorentz transformations. The reason why the ordinary formalism of the quantum field theory is so unsatisfactory lies in the fact that one has built up this theory in the way which is too much analogous to the ordinary non-relativtstic mechanics. In this ordinary formalism of the quantum theory of fields the theory is devided into two distinct sections: the section giving the kinematical relations between various quantities at the same instant of time, and the section determining the causal relations between quantities at different instants of time. Thus the commutation relations (1) belong to the first section and the Schrodinger equation (2) to the second.

As stated defore, this way of separating the theory into two sections is very unrelativistic, since here the concept "same instant of time" plays a distinct rôle.

Also in our formalism the theory is devided into two sections. But now the separation is introduced in another place: In our formalism the theory consists of two sections, one of which gives the laws of behavior of the fields when they are left alone, and the other of which gives the laws determining the deviation from this behavior due to interactions. This way of separating the theory can be carried out relativistically.

Although in this way the theory can be brought into more satisfactory form, no new contents are added thereby. So, the well known divergence difficulties of the theory are inherited also by our theory. Indeed, our fundamental equations (29) admit only catastrophal solutions as can be seen directly in the fact that the unavoidable infinity due to non-vanishing zero-point amplitudes of the fields inheres in the operator $H_{12}(P)$. Thus, a more profound modification of the theory is required in order to remove this fundamental difficulty.

It is expected that such a modification of the theory would possiblly be introduced by some revision of the concept of interaction, because we meet no such difficulty when we deal with the non-interacting fields. This revision would then result that in the separability of the theory into two sections, one for free fields and one for interactions, some uncertainty would be introduced. This seems to be implied by the very fact that, when we formulate the quantum field theory in a relativistically satisfactory manner, this way of separation has revealed itself as the fundamental element of the theory.

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Some Remarks on Bopp's Field Theory.*

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§ 1. Introduction and Summary.

The convergence difficulty of the self-force of the electron has been inherent in the classical theory, and it is essencially due to the assumption that the source of the field is a geometrical point. Recently, Bopp has proposed a new method to remove this difficully by taking the Lagrange-function which involves the higher space-time derivatives of the field quantities.⁽¹⁾ For example, we would obtain the finite field energy, provided that the scalar potential φ satisfies the following equation in the presence of a point singularity resting at the origin,

$$\varDelta(\varDelta-\varkappa^2)\varphi=-4\pi e\delta(\vec{x}). \qquad (1)$$

Because φ is given by the suitable linear combination of the Coulomb potential and Yukawa potential having the common source at the origin so as to be finite up to the singularity.

When Lagrange-function contains the second time-derivatives of the field quantities as is in Bopp's case, the field equations are 4-th order with respect to the time differentiation. So we cannot use the ordinary method of field quantization. Nevertheless, we can show that the quantization of such a field is formally possible by the analogous method to the well-known Heisenberg-Pauli s one — what is an aim of this paper.

As a result of this quantization, it becomes clear that the field is made up of two kinds of Bose particles, one of which has the positive energy but another the negative energy. Therefore the total field energy is never positive definite. Moreover, the system cannot form any stable state, when the interaction with the singularity is introduced. This is the greatest difficulty of Bopp's theory. But, the self-energy of the total system still remains finite in the quantized form of the theory as in the classical one. Of course, the zero-point energy — which is a quantum effect — is present and its absolute value is infinite.

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Here we take the unitaristic standpoint of view as in original Bopp's theory. So, for example, we consider the coordinate of the singular point only as a simple parameter, and describe its motion by the classical equation derived from Born's principle,⁽²⁾ which is not of the canonical form. Therefore the Hamilton-function of the field does not coincide with the total energy of the system. Finally we pick up such a part of energy that is only dependent upon the coordinates of the singularity and show that it takes the form of the energy expression of a particle with a negative mass, when making a uniform motion with a small velocity.

§ 2. Classical treatment.

As the convergence of the self-energy in Bopp's theory is essencially due to the form of the equation like (1), we can discuss the general properties of this theory by taking the scalar theory for simplicity. By using, for convenience, the following units and notations:

$$c=1,\ \hbar=1,\quad (x,\ y,\ z,\ it)\equiv(x_\alpha) \tag{2}$$

and letting $\psi(x_\alpha)$ be the scalar potential and

$$f_\alpha=\frac{\partial\psi}{\partial x_\alpha} \tag{3}$$

the strength of the field, we define the Lagrange-function as follows:

$$L=-\frac{1}{2}\left(f_\alpha^2+\frac{1}{\varkappa^2}\left(\frac{\partial f_\alpha}{\partial x_\beta}\right)^2\right)+\rho\psi. \tag{4}$$

Then we can derive the field equations from the following variation principle:

$$\delta\int L\,dx\,dy\,dz\,dt=0 \tag{5}$$

where $\psi(x_\alpha)$ variates independently, and get

$$\frac{\partial F_\alpha}{\partial x_\alpha}=-\rho,\quad F_\alpha=-\frac{\delta\overline{L}}{\delta f_\alpha}=f_\alpha-\frac{1}{\varkappa^2}\Box f_\alpha, \tag{6}$$

or

$$\Box\left(1-\frac{1}{\varkappa^2}\Box\right)\psi=-\rho. \tag{6'}$$

We can express this equation in the form:

$$\frac{\partial T_{\alpha\beta}}{\partial x_\beta}=-\rho f_\alpha \tag{6''}$$

where $T_{\alpha\beta}(=T_{\beta\alpha})$ is the energy-momentum tensor of the field, and its components are given by

$$\left.\begin{aligned} T_{\alpha\beta}&=\left(F_\alpha F_\beta-\frac{1}{2}\delta_{\alpha\beta}F_\gamma^2\right)-\left(u_\alpha u_\beta-\frac{1}{2}\delta_{\alpha\beta}(u_\gamma^2+u^2)\right), \\ u_\alpha&=F_\alpha-f_\alpha=-\frac{1}{\varkappa^2}\Box f_\alpha, \quad u=\frac{1}{\varkappa}\frac{\partial f_\alpha}{\partial x_\alpha}=\frac{1}{\varkappa}\Box\psi. \end{aligned}\right\}. \tag{7}$$

Above all, for $\alpha=4$ equation (6″) gives the following continuity equation

$$\left.\begin{aligned} &\operatorname{div}\vec{S}+\frac{\partial W}{\partial t}=\rho\frac{\partial\psi}{\partial t}, \\ &W=-T_{44}, \quad S_k=-iT_{4k}. \end{aligned}\right\}. \tag{8}$$

So W is the energy density of the field. Using this expression, we can compute the total energy of the field produced by a singular charge resting at the point $\vec{x}_0$ and obtain the finite result

$$E=\int W\,dx\,dy\,dz=2\pi\varkappa e^2. \tag{9}$$

In order to complete Bopp's theory, it is necessary to derive the equations of motion of the point singularities from an unitaristic standpoint of view. For this purpose, substituting the solution of field equations (6) into

$$L=-\frac{1}{2}\left\{f_\alpha F_\alpha+\frac{1}{\varkappa^2}\frac{\partial}{\partial x_\beta}\left(f_\alpha\frac{\partial f_\alpha}{\partial x_\beta}\right)\right\}+\rho\psi$$

and subtracting the complete time-differential terms, we get

$$M=\overline{L}-\frac{d\Delta}{dt}=\int\left(\frac{1}{2}\psi\frac{\partial F_\alpha}{\partial x_\alpha}+\rho\psi\right)dx\,dy\,dz=\frac{1}{2}\int(\rho\psi)\,dx\,dy\,dz. \tag{10}$$

When there are many singularities, ρ is given by the following expression:

$$\rho=\sum_{\nu}\rho_{\nu}=4\pi\sum_{\nu}e_{\nu}\sqrt{1-V_{\nu}^{2}}\,\delta(\vec{x},\ \vec{x}_{\nu}),\quad \vec{V}_{\nu}=\frac{d\vec{x}_{\nu}}{dt}$$

$$M=2\pi\sum_{\nu}e_{\nu}\sqrt{1-V_{\nu}^{2}}\,\psi(\vec{x}_{\nu}). \tag{11}$$

We obtain, variating $\vec{x}_{\nu}$'s independently,

$$\delta\int M\,dt=\sum_{\nu}\int(\vec{K}_{\nu}\cdot\delta\vec{x}_{\nu})\,dt=0. \tag{12}$$

It is just Born's principle to assume

$$\vec{K}_{\nu}=0 \tag{13}$$

for the equation of motion of the ν-th singularity. In fact, according to Born or Bopp, it is easily shown that (11) gives the ordinary Lagrangian function for many particles each with finite mass and moving in the field produced by other particles, when their accelerations are all small.

Hereafter we take the simplest case of one singularity and denote its coördinate by $\vec{x}_{0}$. The eq. (6′) becomes

$$\Box(\Box-\varkappa^{2})\psi=4\pi e\varkappa^{2}\sqrt{1-V^{2}}\,\delta(\vec{x}-\vec{x}_{0}),\quad \vec{V}=\frac{d\vec{x}_{0}}{dt}.$$

Expanding ψ within a unit cube with periodic condition,

$$\psi=\sum_{\vec{k}}a(k,\ t)e^{i\vec{k}\vec{x}}$$

we get

$$\left(\frac{d^{2}}{dt^{2}}+k^{2}\right)\left(\frac{d^{2}}{dt^{2}}+\omega_{k}^{2}\right)a(k,\ t)=4\pi e\varkappa^{2}\sqrt{1-V^{2}}\,e^{-i\vec{k}\vec{x}_{0}},\quad \omega_{k}=\sqrt{k^{2}+\varkappa^{2}}\,.$$

Then, we solve these equations in following forms

$$a(k,\ t)=\int A(k,\ \nu)e^{-i\nu t}\,d\nu,$$

$$A(k,\ \nu)=2c\varkappa^{2}\int\sqrt{1-V'^{2}}\,e^{i\nu t'-i\vec{k}\vec{x}_{0}'}\,dt'/(\nu^{2}-k^{2})(\nu^{2}-\omega_{k}^{2});$$

$$x_{0}'\equiv x_{0}(t'),\quad \vec{V}'=\vec{V}(t')$$

$$a(k,\ t)=2e\varkappa^{2}\int_{-\infty}^{\infty}\sqrt{1-V'^{2}}\,e^{-i\vec{k}\vec{x}_{0}'}\,dt'\,a(t,\ t')$$

where $$a(t,\ t')=\int_{-\infty}^{\infty} d\nu e^{i\nu(t'-t)}/(\nu^2-k^2)(\nu^2-\omega_k^2).$$

In this place there are two possible pathes for integral, namely (1) the retarded case (Fig. 1); where

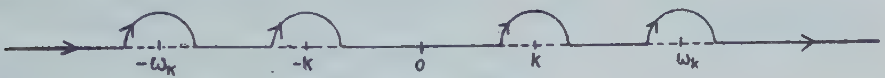

Fig. 1.

$$a^{(1)}(t,\ t')=\begin{cases}\dfrac{2\pi}{\varkappa^2}\left(\dfrac{\sin k(t-t')}{k}-\dfrac{\sin \omega_k(t-t')}{\omega_k}\right) & t\geq t'\\ 0 & t<t'\end{cases}$$

$$a^{(1)}(k,\ t)=4\pi e\int_{-\infty}^{t} dt'\sqrt{1-V'^2}\, e^{-i\vec{k}\vec{x}_0'}\left(\frac{\sin k(t-t')}{k}-\frac{\sin \omega_k(t-t')}{\omega_k}\right)$$

and (2) the advanced case (Fig. 2), where

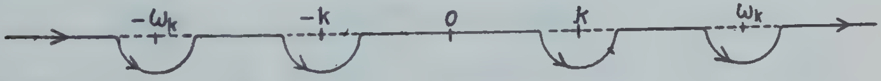

Fig. 2.

$$a^{(2)}(t,\ t')=\begin{cases}0 & t>t'\\ \dfrac{2\pi}{\varkappa^2}\left(\dfrac{\sin k(t'-t)}{k}-\dfrac{\sin \omega_k(t'-t)}{\omega_k}\right) & t\leqq t',\end{cases}$$

$$a^{(2)}(k,\ t)=4\pi e\int_{t}^{\infty} dt'\sqrt{1-V'^2}\, e^{-i\vec{k}\vec{x}_0'}\left(\frac{\sin k(t'-t)}{k}-\frac{\sin \omega_k(t'-t)}{\omega_k}\right)$$

Classifying all quantities corresponding to above two cases by the suffices (1) and (2), we compute the expressions $\int M^{(1)}dt$ and $\int M^{(2)}dt$, which are proved to be equal.

$$\int M^{(1)}dt=\int M^{(2)}dt.$$

After some calculations we find

$$\vec{K}_0=2\pi\left\{\sqrt{1-V^2}\ \mathrm{grad}_{x_0}\left(\psi^{(1)}(\vec{x}_0, t)+\psi^{(2)}(\vec{x}_0, t)\right)\right.$$
$$\left.+\frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-V^2}}\left(\psi^{(1)}(\vec{x}_0, t)+\psi^{(2)}(\vec{x}_0, t)\right)\right)\right\}.$$

So according to Born's principle, the equation of motion for the point singularity becomes

$$\frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-V^2}}(-2\pi e)\left(\psi^{(1)}(\vec{x}_0, t)+\psi^{(2)}(\vec{x}_0, t)\right)\right)$$
$$=-\sqrt{1-V^2}\ \mathrm{grad}_{x_0}(-2\pi e)\left(\psi^{(1)}(\vec{x}_0, t)+\psi^{(2)}(\vec{x}_0, t)\right). \quad (13)$$

Bopp has shown that Born's equation of motion does not contain the effect of radiation reaction. Exchange of energy and momentum between field and singularity is always reversible. Hence, to take this effect into consideration, we should replace eq. (13) by the following equation

$$\frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-V^2}}(-4\pi e)\ \psi^{(1)}(\vec{x}_0, t)\right)$$
$$=-\sqrt{1-V^2}\ \mathrm{grad}_{x_0}(-4\pi e)\ \psi^{(1)}(\vec{x}_0, t). \quad (13')$$

(13) or (13′) is clearly relativistic invariant. For, we can get the energy equation by taking the scalar-product with $\vec{V}$:

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t)\right)=\sqrt{1-V^2}\frac{\partial}{\partial t}\Psi(\vec{x}_0, t) \quad (14)$$

where $\Psi=\begin{cases}-2\pi e(\psi^{(1)}+\psi^{(2)}) & \text{in Born's case,}\\ -4\pi e\ \psi^{(1)} & \text{in Bopp's case,}\end{cases}$

and sum up above four equations in the following tensor form.

$$\frac{d}{d\tau}\left(\frac{dx_\alpha}{d\tau}\Psi(\vec{x}_0, t)\right)=-\frac{\partial}{\partial x_\alpha}\Psi(\vec{x}_0, t) \quad (15)$$

where τ is the proper time of this singularity. From eq. (14) and eq. (8), we get

$$\mathrm{div}\ \vec{S}+\frac{\partial W}{\partial t}=-\delta(\vec{x}, \vec{x}_0)\frac{d}{dt}\left(\frac{1}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t)\right).$$

$$\therefore \quad \frac{d}{dt}\Big[\int W\, dx\, dy\, dz + \frac{1}{\sqrt{1-V^2}}\Psi(\overrightarrow{x_0},\ t)\Big]=0.$$

Therefore, we can interprete $\frac{1}{\sqrt{1-V^2}}\Psi(\overrightarrow{x_0},\ t)$ as the energy concentrated on the singularity. That Born's equation of motion does not contain the radiation reaction can be infered from the fact that we get the following relation by integrating (14) with respect to time:

$$\left[\frac{1}{\sqrt{1-V^2}}\Psi(\overrightarrow{x_0},\ t)\right]_{-\infty}^{\infty}=\int_{-\infty}^{\infty}dt\, \sqrt{1-V^2}\,\frac{\partial\Psi}{\partial t}$$

$$=\begin{cases} 0 & \text{in Born's case} \\ \displaystyle\int_{-\infty}^{\infty}dt\, \sqrt{1-V^2}\,\frac{\partial}{\partial t}\Big\{(-2\pi e)\,(\psi^{(1)}-\psi^{(2)})\Big\} & \text{in Bopp's case.} \end{cases}$$

Next, We transform the field equations into the Hamiltonian form before we perform the quantization. Writing L in a slightly different form form (4),

$$L=-\frac{1}{2}\Big(f_i f_i-\dot{\psi}^2+\frac{1}{\varkappa^2}\frac{\partial f_i}{\partial x_k}\frac{\partial f_i}{\partial x_k}-\frac{2}{\varkappa^2}\frac{\partial\dot{\psi}}{\partial x_k}\frac{\partial\dot{\psi}}{\partial x_k}+\frac{1}{\varkappa^2}\ddot{\psi}^2\Big)+\rho\psi,$$

we introduce the following variables:

$$\left.\begin{aligned} \psi^+&=\frac{\delta\overline{L}}{\delta\dot{\psi}}-\frac{d}{dt}\Big(\frac{\delta\overline{L}}{\delta\ddot{\psi}}\Big)=\dot{\psi}-\frac{2}{\varkappa^2}\Delta\dot{\psi}+\frac{1}{\varkappa^2}\dddot{\psi} \\ \chi&=\dot{\psi} \\ \chi^+&=\frac{\delta\overline{L}}{\delta\ddot{\psi}}=-\frac{1}{\varkappa^2}\ddot{\psi} \end{aligned}\right\}\qquad(16)$$

The Hamiltonian of the field is defined to be

$$H=\psi^+\chi+\chi^+\ddot{\psi}-L=\psi^+\chi+\frac{1}{2}\Big(f_i f_i-\chi^2$$
$$+\frac{1}{\varkappa^2}\frac{\partial f_i}{\partial x_k}\frac{\partial f_i}{\partial x_k}-\frac{2}{\varkappa^2}\frac{\partial\chi}{\partial x_i}\frac{\partial\chi}{\partial x_i}-\varkappa^2\chi^{+2}\Big)-\rho\psi.\qquad(17)$$

If we make the canonical equations by taking $(\psi,\ \psi^+)$, $(\chi,\ \chi^+)$ as canonical variables,

$$\left.\begin{aligned}
\dot{\psi} &= \frac{\delta \overline{H}}{\delta \psi^+} = \chi \\
\dot{\psi}^+ &= -\frac{\delta \overline{H}}{\delta \psi} = \rho + \Delta\psi - \frac{1}{\kappa^2}\Delta\Delta\psi \\
\chi &= \frac{\delta \overline{H}}{\delta \chi^+} = -\kappa^2 \chi^+ \\
\dot{\chi}^+ &= -\frac{\delta \overline{H}}{\delta \chi} = -\psi^+ + \chi - \frac{2}{\kappa^2}\Delta\chi
\end{aligned}\right\} \qquad (18)$$

it is easily shown that (18) coincides with (16) and (6′). The relation between $\overline{H}$ and $E=\int W\,dx\,dy\,dz$ becomes then

$$\overline{H} = E - \int \rho\psi\, dx\,dy\,dz. \qquad (19)$$

§ 3. Quantization and the Self-energy.

Having found the Hamiltonian form, quantization is simple, We put, according to Heisenberg and Pauli, the following commutation rules between the field quantities:

$$\left.\begin{aligned}
[\psi(\vec{x},\,t),\ \psi^+(\vec{x}',\,t)] &= i\delta(\vec{x},\,\vec{x}') \\
[\chi(\vec{x},\,t),\ \chi^+(\vec{x}',\,t)] &= i\delta(\vec{x},\,\vec{x}')
\end{aligned}\right\} \qquad (20)$$

all other pairs of quantities being taken to be commutable. If we postulate the equations of motion for ψ, ψ^+, χ, χ^+ to be

$$i\dot{\psi} = [\psi,\ \overline{H}] \quad \text{etc.,} \qquad (21)$$

it is well known that the classical equations (18) can be derived from (20) and (21). Hence, for a functional $G(\psi\ldots,\,t)$ containing the time t explicitly we can put, generally,

$$iG = i\frac{\partial G}{\partial t} + [G,\ \overline{H}] \qquad (22)$$

Particularly, for the Hamiltonian $\overline{H}$

$$\frac{d\overline{H}}{dt} = \frac{\partial \overline{H}}{\partial t} = -\int \frac{d\rho}{dt}\psi\, dx\,dy\,dz$$

so that $\frac{d\bar{H}}{dt} \neq 0$ in general. We cannot, therefore, consider the Hamiltonian of the field to be the energy of total system, because it is not a constant of motion. Only when $\frac{d\rho}{dt}=0$, especially when the singularity is at rest, $\bar{H}$ represents the energy of the system.

Here we take the standpoint that only the field quantities are to be quantized (q-numbers !), and the coordinates of the sigularity are mere c-numbers. Since the field equations describe the feature of the field when the motion of the singularity has been given, we must add, to complete the problem, the equation which determines the motion of the singularity (though it is only a parameter) and is compatible with the field equations. It is not necessary to write this equation in canonical form. One of the simplest way is to borrow the classical equation

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\vec{V}}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t)\right) = -\sqrt{1-V^2}\ \mathrm{grad}_{x_0}\ \Psi(\vec{x}_0, t) \\ &\text{with}\quad \Psi(\vec{x}_0, t) = -4\pi e\ \psi(\vec{x}_0, t). \end{aligned}\right\} \qquad (23)$$

We have in this case

$$\frac{d\bar{H}}{dt} = -\frac{d}{dt}\left(\frac{V^2}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t)\right)$$

Hence,

$$\frac{d\mathbf{H}}{dt}=0, \quad \mathbf{H} \equiv \bar{H} + \frac{V^2}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t) = E + \frac{1}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t) \quad (24)$$

H, which may be called the energy of the system, consists of two parts. We may consider $\frac{1}{\sqrt{1-V^2}}\Psi(\vec{x}_0, t)$ as the energy concentrated on the singularity, while E may be called the energy of the field.

To see the particle character of the field, it is convenient to go to the momentum space. We put, namely,

$$\left.\begin{aligned} &\psi = \sum_{\vec{k}} q(\vec{k})\, e^{i\vec{k}\vec{x}}, \qquad \psi^{+} = \sum_{\vec{k}} p(\vec{k})\, e^{-i\vec{k}\vec{x}}, \\ &\chi = \sum_{\vec{k}} Q(\vec{k})\, e^{i\vec{k}\vec{x}}, \qquad \chi^{+} = \sum_{\vec{k}} P(\vec{k})\, e^{-i\vec{k}\vec{x}} \end{aligned}\right\} \qquad (25)$$

Then we get the following commutation relations :

$$[q(k),\ p(k')]=[Q(k),\ P(k')]=i\delta_{kk'} \tag{25a}$$

while all the other pairs are commutable. Further, we perform another transformation of variables, putting.

$$\left.\begin{aligned}
q(\vec{k})&=\frac{1}{\sqrt{2k}}\Big(A(-\vec{k})+A^*(\vec{k})\Big)+\frac{1}{\sqrt{2\omega_k}}\Big(B(\vec{k})+B^*(-\vec{k})\Big)\\
p(-\vec{k})&=-\frac{ik\omega_k}{\varkappa^2}\left\{\frac{\omega_k}{\sqrt{2k}}\Big(A(-\vec{k})-A^*(\vec{k})\Big)-\frac{k}{\sqrt{2\omega_k}}\Big(B(\vec{k})-B^*(-\vec{k})\Big)\right\}\\
Q(\vec{k})&=i\sqrt{\frac{k}{2}}\Big(A(-\vec{k})-A^*(\vec{k})\Big)+i\sqrt{\frac{\omega_k}{2}}\Big(B(\vec{k})-B^*(-\vec{k})\Big)\\
P(-\vec{k})&=\frac{1}{\varkappa^2}\left\{k^2\Big(A(-\vec{k})+A^*(\vec{k})\Big)+\omega_k^2\Big(B(\vec{k})+B^*(-\vec{k})\Big)\right\}
\end{aligned}\right\} \tag{26}$$

In these variables we find the commutaion rules to be

$$[A(\vec{k}),\ A^*(\vec{k}')]=[B(\vec{k}),\ B^*(\vec{k}')]=i\delta_{kk'} \tag{26a}$$

all the other pairs being commutable. For the energy of the system we obtain, in terms of A, A^*, B and B^*,

$$\mathbf{H}=\overline{H}_0+\overline{H}' \tag{27}$$

with $\overline{H}_0=\sum\limits_{\vec{k}}\left\{k\Big(A^*(\vec{k})A(\vec{k})+\frac{1}{2}\Big)-\omega_k\Big(B^*(\vec{k})B(\vec{k})+\frac{1}{2}\Big)\right\}$ (27a)

$$\begin{aligned}
\overline{H}'=-\frac{4\pi e}{\sqrt{1-V^2}}\sum_{\vec{k}}\Bigg\{&e^{i\vec{k}\vec{x}_0}\left(\frac{A^*(\vec{k})}{\sqrt{2k}}+\frac{B(\vec{k})}{\sqrt{2\omega_k}}\right)\\
&+e^{-i\vec{k}\vec{x}_0}\left(\frac{A(\vec{k})}{\sqrt{2k}}+\frac{B^*(\vec{k})}{\sqrt{2\omega_k}}\right)\Bigg\}
\end{aligned} \tag{27b}$$

As a consequence of (26a) the operators

$$N_A(\vec{k})=A^*(\vec{k})A(\vec{k}) \quad \text{and} \quad N_B(\vec{k})=B^*(\vec{k})B(\vec{k})$$

has all positive integers and 0 as proper values. Thus, we know that the free field $\mathbf{H}=\overline{H}_0$ consists of two kinds of particles. This is the result of the fact that our Lagrangian contains the second order derivatives of the field variables with respect to time. Instead of the initial conditions in the classical theory, we may take here the initial values of the complete set of observables such as, say, $N_A(\vec{k})$, $N_B(\vec{k})$.

To get the self-energy of the system it is convenient to make the following canonical transformations:

$$\left.\begin{aligned} A(\vec{k}) &= A'(\vec{k}) + \frac{4\pi e}{\sqrt{1-V^2}} e^{i\vec{k}\vec{x}_0} / k\sqrt{2k} \\ B(\vec{k}) &= B'(\vec{k}) - \frac{4\pi e}{\sqrt{1-V^2}} e^{-i\vec{k}\vec{x}_0} / \omega_k\sqrt{2\omega_k} \end{aligned}\right\} \qquad (28)$$

Then we have

$$\left.\begin{aligned} &\mathbf{H} = \sum_{\vec{k}} \left\{ k\left(A'^*(\vec{k})A'(\vec{k}) + \frac{1}{2}\right) - \omega_k\left(B'^*(\vec{k})B'(\vec{k}) + \frac{1}{2}\right)\right\} + \Delta E \\ &\text{where} \qquad \Delta E = -2\pi\varkappa e^2/(1-V^2). \end{aligned}\right\} \qquad (29)$$

$\mathbf{H}-\Delta E$ has the same spectrum as the Hamiltonian of the free field. ΔE represents, therefore; the energy-shift caused by the presence of the singularity, and can be interpreted as the self-energy of the system. Thus we see that the finiteness of the self-energy in classical Bopp's theory is kept also when we go into the quantum theory. Particularly, when the singularity is at rest, we find

$$\Delta E = -2\pi\varkappa e^2 \qquad (9')$$

which has the opposite sign to (9). This corresponds to the circumstance that we have calculated in (9) the energy of the field only — in the classical theory the self-energy is always interpreted in this meaning —, while in (9′) we have calculated the energy of the system consisting of the field and the singularity. Further, we should remark the fact that ΔE are not transformed as the time-component of a four-vector when the velocity of the singularity varies.

$\overline{H}_0$ contains the zero-point energy whose absolute value is infinity. But the more serious difficulty in Bopp's theory is the appearance of negative energy particles in $\overline{H}_0$. In consequence, $\mathbf{H}$ is not positive definite and it is impossible to save this difficulty by taking account of the hole theory, since the particles of our problem obey Bose statistics. The system, therefore, can not form the stable state even if we neglect the zero-point energy. It seems for us that the success of Bopp's theory is to be confined to the classical region.

§ 4. Semi-classical correspondence.

In the previous section we have separated **H** into $\overline{H}_0$ and $\overline{H}'$. Another separation is possible which will now be made. $A(\vec{k})$, as it satisfies the following equation

$$i\dot{A}(\vec{k})=[A(\vec{k}),\ \overline{H}]=kA(\vec{k})-\frac{4\pi e}{\sqrt{1-V^2}}e^{i\vec{k}\vec{x}_0}/\sqrt{2k}$$

consists of two parts:

$$A(\vec{k})=A_0(\vec{k})+u(\vec{k})$$

where $A_0(\vec{k})$ is the solution of the homogeneous equation made from above one corresponding to the free field, and $u(\vec{k})$ is the non-homogeneous solution of this equation, namely:

$$u(\vec{k})=-\frac{4\pi ei}{\sqrt{2k}}\int_{-\infty}^{t}dt'e^{i\vec{k}\vec{x}_0'+ik(t-t')}/\sqrt{1-V'^2}.$$

Similarly we can write for $B(\vec{k})$

$$B(\vec{k})=B_0(\vec{k})+v(\vec{k})$$

with
$$v(\vec{k})=-\frac{4\pi ei}{\sqrt{2\omega_k}}\int_{-\infty}^{t}dt'e^{-i\vec{k}\vec{x}_0'-i\omega_k(t-t')}/\sqrt{1-V'^2}.$$

Here we can consider u, v as c-numbers. Inserting these expressions into (27) we obtain

$$\mathbf{H}=\overline{H}^0+\overline{H}^i+\overline{H}^s \tag{30}$$

with

$$\overline{H}^0=\sum_{\vec{k}}\left\{k\left(A_0^*(\vec{k})A_0(\vec{k})+\frac{1}{2}\right)-\omega_k\left(B_0^*(\vec{k})B_0(\vec{k})+\frac{1}{2}\right)\right\} \tag{30a}$$

$$\overline{H}^s=\sum(ku^*(\vec{k})u(\vec{k})-\omega_k v^*(\vec{k})v(\vec{k}))$$

$$-\frac{4\pi e}{\sqrt{1-V^2}}\sum_{\vec{k}}\left\{e^{i\vec{k}\vec{x}_0}\left(\frac{u^*(\vec{k})}{\sqrt{2k}}+\frac{v(\vec{k})}{\sqrt{2\omega_k}}\right)+e^{-i\vec{k}\vec{x}_0}\left(\frac{u(\vec{k})}{\sqrt{2k}}+\frac{v^*(\vec{k})}{\sqrt{2\omega_k}}\right)\right\} \tag{30b}$$

$\bar{H}^0$ corresponds to the energy of the free field, and $\bar{H}^s$ is a function of the coordinates of the singularity only, while $\bar{H}^i$ may be interpreted as the interaction term. Especially when the singularity is moving uniformly with small velocity $\vec{V}$, $\bar{H}^s$ becomes simply as follows,

$$\bar{H}^s \cong -2\pi\varkappa e^2/\sqrt{1-V^2} \tag{31}$$

so that, in this case $\bar{H}^s$ takes the form as if it were the energy of the particle with negative mass.

In conclusion we should like to express our deep gratitude to Professor M. Kobayasi for his kind interest and encouragement in this work. Further we are indebted to him and Mr. S. Kokaji for their valuable discussions.

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On the Interaction between Vector Mesons and Nucleons.*

By Shoji OZAKI

(Recievcd May 21, 1946.)

It was shown by several authors that the perturbation theory, which proved in the case of quantum electrodynamics to be so powerfull a method of attack, was not adequate for the problems concerning the interaction of mesons with nucleons, firstly because their interaction constant $g^2/\hbar c$ is not small enough in contrast to $e^2/\hbar c$ which describes the interaction of electrons with electromagnetic field, and secondly the proper mesonic field around a nucleon is of dipole type having higher singularities as compared to the proper electromagnetic field around an electron. Thus the adequate method of approximation for the strong interaction between mesons and nucleons has been set forth by Wentzel,⁽¹⁾ and for the case of moderate as well as strong coupling the method, which resembles closely to the Hartree approximation for the ordinary atomic system, has been proposed by S. Tomonaga.⁽²⁾ By using the latter method, detailed investigations were done for charged longitudinal, symmetrical longitudinal and neutral pseudoscalar mesons interacting with nucleons.⁽³⁾ In the present paper we will now investigate the problem about the interaction of nucleons with charged vector mesons, taking into account not only the longitudinal but also the transverse part in order to know the complete natures of the interaction. To simplify the problem, it is also here assumed, as has been done by Wentzel and Tomonaga, that the nucleon is infinitely heavy. Then the interaction energies between vector mesons and nucleons are separated into two parts, the longitudinal and the transverse part and as to the latter part, it can be shown that only the transverse mesons, whose orbital angular momenta are perpendicular to the spins, i. e. $l=j$ which we will call c-component of the mesons interact with nucleons.

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* The content of this paper was read at the Symposium of the National Research Council of Japan, on Nov. 18, 1944.

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(2) S. Tomonaga; Sc. Pap. I.P.C.R., **39** (1941), 247.

(3) loc. cit. (2); T. Miyazima and S. Tomonaga; Sc. Pap. I.P.C.R., **40** (1942), 21; S. Tomonaga; ibid, **40** (1942), 73; T. Miyazima and S. Tomonaga; ibid. **40** (1943), 274; S. Tomonaga; Rikagakukenkyujo Iho. **22** (1943), 711 & 781.

The eigenvalue problem for the nucleon interacting with charged vector mesons is thus solved in the approximate manner described by Tomonaga for the lowest state of the system. The interaction energies of c-component of the transverse vector mesons and a nucleon are the same as those in the case of the charged pseudoscalar mesons only differing in the interaction parameters, $l'=\frac{1}{2\pi}\frac{g_2}{\sqrt{\hbar c}}\frac{1}{\varkappa}\sqrt{\frac{2}{3}}$ ($\varkappa$ denotes the reciprocal of Compton wave length of a meson) for the former and $l''=\frac{1}{2\pi}\frac{f_2}{\sqrt{\hbar c}}\frac{1}{\varkappa}\frac{1}{\sqrt{3}}$ for the latter. g_2 and f_2 are the interaction constants introduced by H. Yukawa(4) and Y. Tanikawa and H. Yukawa(5) respectively. The allied problems for the charged pseudoscalar meson were treated by S. Kanesawa, H. Jen and S. Tomonaga.(6) The self-energy of the proton due to c-component of the transverse charged vector meson is, as known from the above fact, of the same magnitude as that due to the pseudoscalar charged meson if the interaction constants l' and l'' are put as equal, and this value for strong coupling is 1/3 times so small as compared to the value obtained by the perturbation theory. For the case of moderate coupling, though much interesting, the calculation can not yet be given owing to the complexity. The anomalous magnetic moment of the proton is also estimated, calculating the mean value of the interaction energy between the mesons and the applied magnetic field for the lowest state of the system using the approximate wave functions obtained. The magnitude of the anomalous magnetic moment thus obtained is about 1/14 times so small as that calculated according to the ordinary perturbation theory,(7) if one takes the same cutting-off momentum in both cases. If the cutting-off momentum of the diverging integral and the interaction constant $g_2/\sqrt{\hbar c}$ are assumed as $10\varkappa$ and 0.3 respectively, the former of which is determined from the observed value of the experimental cross sections of scattering of mesons by nucleons, the perturbational method gives for the anomalous magnetic moment about $20\mu_0$ (μ_0 being a nuclear magneton), which is about ten times too large as compared to the experimental value. Our calculation shows for strong coupling that the anomalous magnetic moment becomes about $1.4\mu_0$ and thus is of the right oder. The comparison with the experimental value suggests that the interaction of nucleons with mesons is rather moderate. It is noted that the value given by the charged pseudoscalar meson theory is as half times as that by the charged vector meson theory, in both cases of strong and weak interaction.

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(4) H. Yukawa, S. Sakata and M. Taketani; Proc. Phys.-Math. Soc. Japan, **20** (1938), 319.
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§ 1

When a nucleon is assumed to be infinitely heavy and rest at the origin, the Hamiltonian of meson field interacting with it may be expressed as follows:

$$\begin{aligned} H=&\frac{1}{4\pi}\sum_{i=1}^{2}\int\left\{\vec{F}_i^2+\frac{1}{\varkappa^2}(\operatorname{div}\vec{F}_i)^2+\frac{1}{\varkappa^2}(\operatorname{rot}\vec{U}_i)^2+\vec{U}_i^2\right\}d\tau\\ &-\frac{1}{\varkappa^2}\sum_{i=1}^{2}\int\left\{g_1\tau_i\operatorname{div}\vec{F}_i-g_2\tau_i(\vec{\sigma},\operatorname{rot}\vec{U}_i)\right\}\delta(\vec{r})\,d\tau, \qquad (1.1)\end{aligned}$$

where $\vec{F}_i$, $\vec{U}_i$ are the real field variables for the meson and τ_1, τ_2, which appears in the second term expressing the interaction energy of mesons with the static nucleon, are, with τ_3, the three components of charge spin τ of the nucleon, which satisfy the commutation relations:

$$[\tau_1,\tau_2]=2i\tau_3,\quad [\tau_2,\tau_3]=2i\tau_1,\quad [\tau_3,\tau_1]=2i\tau_2,\quad \tau_1^2=\tau_2^2=\tau_3^2=1. \qquad (1.2)$$

As we will deal with the states of mesons around the fixed nucleon, it is convenient to treat the problem by expanding the field variables in spherical coordinates r, θ, φ. We expand the field variables as follows:(8) (9)

$$\begin{aligned}\vec{F}_i^{(a)}=&\sqrt{\pi}\sum_{j,\mu}\int\sqrt{\frac{\varkappa\hbar c}{2j+1}}\sqrt{\frac{\varkappa}{K}}\Big(A_{j,\mu}^{(i)*}(k)-(-1)^{\mu}A_{j,-\mu}^{(i)}(k)\Big)\\ &\times\Big(\sqrt{j}\,Q_{j-1}^{(+)}f_{j-1/2}(kr)+\sqrt{j+1}\,Q_{j+1}^{(-)}f_{j+3/2}(kr)\Big)dk,\\ \vec{U}_i^{(a)}=&-i\sqrt{\pi}\sum_{j,\mu}\int\sqrt{\frac{\varkappa\hbar c}{2j+1}}\sqrt{\frac{K}{\varkappa}}\Big(A_{j,\mu}^{(i)}(k)+(-1)^{\mu}A_{j,-\mu}^{(i)*}(k)\Big)\\ &\times\Big(\sqrt{j}\,Q_{j-1}^{(+)}f_{j-1/2}(kr)+\sqrt{j+1}\,Q_{j+1}^{(-)}f_{j+3/2}(kr)\Big)dk,\\ \vec{F}_i^{(b)}=&\sqrt{\pi}\sum_{j,\mu}\int\sqrt{\frac{\varkappa\hbar c}{2j+1}}\sqrt{\frac{K}{\varkappa}}\Big(B_{j,\mu}^{(i)*}(k)-(-1)^{\mu}B_{j,-\mu}^{(i)}(k)\Big)\end{aligned}$$

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(8) S. Ozaki; Sc. Pap. I.P.C.R. **39** (1941), 223.

(9) S. Ozaki; Nippon Sugaku-Butsurigakkai Kwaishi, **17** (1943), 498.

$$\left.\begin{aligned}
&\qquad\times\left(\sqrt{j+1}\,Q_{j-1}^{(+)}f_{j-1/2}(kr)-\sqrt{j}\;Q_{j+1}^{(-)}f_{j+3/2}(kr)\right)dk,\\
&\overrightarrow{U}_i^{(b)}=-i\sqrt{\pi}\sum_{j,\mu}\int\sqrt{\frac{\varkappa\hbar c}{2j+1}}\sqrt{\frac{\varkappa}{K}}\left(B_{j,\mu}^{(i)}(k)+(-1)^{\mu}B_{j,-\mu}^{(i)*}(k)\right)\\
&\qquad\times\left(\sqrt{j+1}\,Q_{j-1}^{(+)}f_{j-1/2}(kr)-\sqrt{j}\;Q_{j+1}^{(-)}f_{j+3/2}(kr)\right)dk,\\
&\overrightarrow{F}_i^{(c)}=\sqrt{\pi}\sum_{j,\mu}\int\sqrt{K\hbar c}\left(C_{j,\mu}^{(i)*}(k)-(-1)^{\mu}C_{j,-\mu}^{(i)}(k)\right)Q_j^{(0)}f_{j+1/2}(kr)\,dk,\\
&\overrightarrow{U}_i^{(c)}=-i\sqrt{\pi}\sum_{j,\mu}\int\frac{1}{\sqrt{K\hbar c}}\left(C_{j,\mu}^{(i)}(k)+(-1)^{\mu}C_{j,-\mu}^{(i)*}(k)\right)Q_j^{(0)}f_{j+1/2}(kr)\,dk,
\end{aligned}\right\}\quad(1.3)$$

where

$$Q_{j-1}^{(+)}=\begin{vmatrix}\sqrt{\dfrac{(j+\mu-1)(j+\mu)}{(2j-1)2j}}\,Y_{j-1}^{\mu-1}\\[2ex] \sqrt{\dfrac{(j+\mu)(j-\mu)}{(2j-1)j}}\,Y_{j-1}^{\mu}\\[2ex] \sqrt{\dfrac{(j-\mu)(j-\mu-1)}{(2j-1)2j}}\,Y_{j-1}^{\mu+1}\end{vmatrix}\qquad Q_{j+1}^{(-)}=\begin{vmatrix}\sqrt{\dfrac{(j-\mu+2)(j-\mu+1)}{2(j+1)(2j+3)}}\,Y_{j+1}^{\mu-1}\\[2ex] \sqrt{\dfrac{(j+\mu+1)(j-\mu+1)}{(j+1)(2j+3)}}\,Y_{j+1}^{\mu}\\[2ex] \sqrt{\dfrac{(j+\mu+1)(j+\mu+2)}{2(j+1)(2j+3)}}\,Y_{j+1}^{\mu+1}\end{vmatrix}$$

$$Q_j^{(0)}=\begin{vmatrix}\sqrt{\dfrac{(j+\mu)(j-\mu+1)}{2j(j+1)}}\,Y_j^{\mu-1}\\[2ex] \dfrac{\mu}{\sqrt{j(j+1)}}\,Y_j^{\mu}\\[2ex] \sqrt{\dfrac{(j+\mu+1)(j-\mu)}{2j(j+1)}}\,Y_j^{\mu+1}\end{vmatrix}$$

and $k=p/\hbar c$, $K=E/\hbar c$ and $\varkappa=m_u c/\hbar$ are the momentum, energy and rest energy of a meson in natural units respectively. $Y_j^{\mu}(\theta,\varphi)$ expresses normalized surface spherical harmonics, j and μ being the quantum numbers corresponding to the total angular momentum and its component along the z-axis of meson respectivery.

$$f_{j+1/2}(kr)=\frac{k}{\sqrt{kr}}J_{j+1/2}(kr)$$

The three pairs of field variables $\overrightarrow{F}_i^{(a)}$, $\overrightarrow{U}_i^{(a)}$; $\overrightarrow{F}_i^{(b)}$, $\overrightarrow{U}_i^{(b)}$; $\overrightarrow{F}_i^{(c)}$, $\overrightarrow{U}_i^{(c)}$ given by (1. 3) satisfy the following relations respectively :

$$\begin{aligned}&\operatorname{rot}\overrightarrow{F}_i^{(a)}=0,\quad\operatorname{rot}\overrightarrow{U}_i^{(a)}=0,\quad\operatorname{div}\overrightarrow{F}_i^{(b)}=0,\quad\operatorname{div}\overrightarrow{U}_i^{(b)}=0,\\&\qquad\qquad\qquad\qquad\qquad\operatorname{div}\overrightarrow{F}_i^{(c)}=0,\quad\operatorname{div}\overrightarrow{U}_i^{(c)}=0.\qquad(1.4)\end{aligned}$$

and therefore $\vec{F}_i^{(a)}$, $\vec{U}_i^{(a)}$ express the longitudinal, $\vec{F}_i^{(b)}$, $\vec{U}_i^{(b)}$; $\vec{F}_i^{(c)}$, $\vec{U}_i^{(c)}$ the two transverse components of the meson field. The latter components correspond to those which Heitler$^{(10)}$ obtained for the electromagnetic field.

The total charge of meson field $q=\int \rho d\tau$, where the charge density is defined by

$$\rho=\frac{e}{4\pi\varkappa\hbar c}\frac{1}{i}\left\{(\vec{U}^*,\ \vec{F})-(\vec{F}^*,\ \vec{U})\right\} \tag{1.5}$$

become then ;

$$q=e\sum_{i=1}^{2}\int\sum_{j,\mu}dk\left\{A_{j,\mu}^{(i)*}(k)A_{j,\mu}^{(i)}(k)+B_{j,\mu}^{(i)*}(k)B_{j,\mu}^{(i)}(k)+C_{j,\mu}^{(i)*}(k)C_{j,\mu}^{(i)}(k)\right\} \tag{1.6}$$

and the Hamiltonian (1. 1) takes the following form :

$$H_0=\hbar c\sum_{i=1}^{2}\sum_{j,\mu}\int dk\ K\left\{A_{j,\mu}^{(i)*}(k)A_{j,\mu}^{(i)}(k)+B_{j,\mu}^{(i)*}(k)B_{j,\mu}^{(i)}(k)+C_{j,\mu}^{(i)*}(k)C_{j,\mu}^{(i)}(k)\right\}. \tag{1.7}$$

On quantizing the meson field, we impose on the amplitudes $A_{j,\mu}^{(i)}(k)$, $B_{j,\mu}^{(i)}(k)$ and $C_{j,\mu}^{(i)}(k)$ the following commutation relations :

$$\begin{aligned}
&[A_{j,\mu}^{(i)}(k),\ A_{j',\mu'}^{(i)*}(k')]=\delta_{jj'}\delta_{\mu\mu'}\delta(k-k')\\
&\qquad[B_{j,\mu}^{(i)}(k),\ B_{j',\mu'}^{(i)*}(k')]=\delta_{jj'}\delta_{\mu\mu'}\delta(k-k'),\\
&\qquad\qquad[C_{j,\mu}^{(i)}(k),\ C_{j',\mu'}^{(i)*}(k')]=\delta_{jj'}\delta_{\mu\mu'}\delta(k-k').
\end{aligned} \tag{1.8}$$

and all other commutators vanish.

The interaction energy can be now reduced to a simple form : in our approximation of the static nucleon only S waves of the longitudinal mesons and of the transverse mesons only P waves of that component whose orbital angular momenta are perpendicular to the spins, i.e. $l=j$, interact with the nucleon. Reserving in H_0 (1. 7) only the corresponding terms, the total Hamiltonian becomes :

$$\begin{aligned}
\frac{H}{\hbar c}&=\sum_{i=1}^{2}\int dk\ K\left\{A_{00}^{(i)*}A_{00}^{(i)}+C_{1,1}^{(i)*}C_{1,1}^{(i)}+C_{1,-1}^{(i)*}C_{1,-1}^{(i)}+C_{1,0}^{(i)*}C_{1,0}^{(i)}\right\}\\
&+l\sqrt{2\pi}\sum_{i=1}^{2}\int dk\,g(k)k(A_{00}^{(i)*}-A_{00}^{(i)})\tau_i-2\sqrt{\pi}\,l'\sum_{i=1}^{2}\int dk\,g(k)k.\\
&\times\left\{(C_{1,1}^{(i)}-C_{1,-1}^{(i)*})\frac{\sigma_x+i\sigma_y}{2}-(C_{1,-1}^{(i)}-C_{1,1}^{(i)*})\frac{\sigma_x-i\sigma_y}{2}-\frac{1}{\sqrt{2}}(C_{1,0}^{(i)*}+C_{1,0}^{(i)})\sigma_z\right\}\tau_i,
\end{aligned} \tag{1.9}$$

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(10) W. Heitler; Proc. Cambr. Phil., Soc. **32** (1936), 112.

where

$$l=\frac{1}{2\pi}\frac{1}{\varkappa}\frac{g_1}{\sqrt{\hbar c}},\quad l'=\frac{1}{2\pi}\sqrt{\frac{2}{3}}\frac{1}{\varkappa}\frac{g_2}{\sqrt{\hbar c}}. \qquad (1.10)$$

In oder to avoid the difficulty of divergence which may be related to the model of a point nucleon, we must introduce in the subsequent considerations the following cutting-off factor:

$$g(k)=\frac{k}{\sqrt{K}}F(k),\qquad \left.\begin{aligned} F(k)&=1 \quad k<\mathrm{K}\\ &=0 \quad k>\mathrm{K}\end{aligned}\right\} \qquad (1.11)$$

K being the cutting-off momentum.

The Hamiltonian (1.9) is now compared with that for the case of charged pseudoscalar mesons, which is in the ordinary representation written as:

$$H=\frac{1}{4\pi}\sum_{i=1}^{2}\int d\tau\left\{P_i^2+(\mathrm{grad}\,Q_i)^2+\varkappa^2Q_i^2-4\pi\frac{f}{\varkappa}\tau_i(\vec{\sigma},\mathrm{grad}\,Q_i)\delta(\vec{r})\right\} \quad (1.12)$$

where Q_i and P_i are real field variables. To proceed to momentum space, we expand them as in the above, into series of surface spherical harmonics:

$$\left.\begin{aligned} Q_i&=\sqrt{\pi}\sum_{l,m}\int\sqrt{\frac{\hbar c}{K}}\left\{A_{l,m}^{(i)}(k)+(-1)^mA_{l,-m}^{(i)*}(k)\right\}Q_{l,m,k}(\vec{r})\,dk,\\ P_i&=i\sqrt{\pi}\sum_{l,m}\int\sqrt{\hbar cK}\left\{A_{l,m}^{(i)*}(k)-(-1)^mA_{l,-m}^{(i)}(k)\right\}Q_{l,m,k}(\vec{r})\,dk\\ Q_{l,m,k}(\vec{r})&=\frac{k}{\sqrt{kr}}J_{l+1/2}(kr)\,Y_l^m(\theta,\varphi).\end{aligned}\right\}. \quad (1.13)$$

The quantized amplitudes $A_{l,m}^{(i)}(k)$ and $A_{l,m}^{(i)*}(k)$ satisfy the commutation relations:

$$[A_{l,m}^{(i)}(k),\ A_{l',m'}^{(i)*}(k')]=\delta_{ll'}\delta_{mm'}\delta(k-k').$$

$$[A_{l,m}^{(i)}(k),\ A_{l',m'}^{(j)}(k')]=[A_{l,m}^{(i)*}(k),\ A_{l',m'}^{(j)*}(k')]=0. \quad (1.14)$$

As easily seen from (1.12) and (1.13), only P waves of charged pseudoscalar mesons interact with static nucleons, so that we consider in Hamiltonian (1.12) the contributions from P waves only. Substituting (1.13) into (1.12) it follows;

$$H=\hbar c\sum_{i=1}^{2}\sum_{l,m}\int KA_{l,m}^{(i)*}(k)A_{l,m}^{(i)}(k)\,dk-2\hbar c\sqrt{\pi}\,l''\sum_{i=1}^{2}\int g(k)k\Big\{(A_{1,1}^{(i)}(k)-A_{1,-1}^{(i)*}(k))\frac{\sigma_x+i\sigma_y}{2}-(A_{1,-1}^{(i)}(k)-A_{1,1}^{(i)*}(k))\frac{\sigma_x-i\sigma_y}{2}-\frac{1}{\sqrt{2}}(A_{1,0}^{(i)*}(k)+A_{1,0}^{(i)}(k))\sigma_z\Big\}\tau_i dk \tag{1. 15}$$

where

$$l''=\frac{1}{2\pi}\frac{1}{\sqrt{3}}\frac{1}{\varkappa}\frac{f}{\sqrt{\hbar c}} \tag{1. 16}$$

This is essentially of the same form as (1. 9) for charged vector mesons, as far as we concern with their transverse c-components, the only difference being that the interaction constant $l''=\frac{1}{2\pi}\frac{1}{\sqrt{3}}\frac{1}{\varkappa}\frac{f}{\sqrt{\hbar c}}$ stands for pseudoscalar mesons instead of $l'=\frac{1}{2\pi}\sqrt{\frac{2}{3}}\frac{1}{\varkappa}\frac{g_2}{\sqrt{\hbar c}}$ for the vector mesons. Accordingly, the eigenvalue problems for both cases

$$(H-E)\Psi=0 \tag{1. 17}$$

are formally identical and we can use the solution obtained by S. Kanesawa, H. Jen and S. Tomonaga$^{(11)}$ for the case of charged pseudoscalar mesons. We cite here the results shortly. We use Hartree approximations for the eigenvalue problem

$$(H-E)\Psi=0$$

expanding $A_{00}^{(i)}$, $C_{1,1}^{(i)}$, $C_{1,-1}^{(i)}$ and $C_{1,0}^{(i)}$ of (1. 9) in the following way:

$$\left.\begin{aligned}
A_{00}^{(i)}&=\frac{\sqrt{4\pi}}{\mathbf{K}_2}\frac{g(k)k}{K}A_0^{(i)}+\sum_{s=1}^{\infty}a_{0s}^{(i)}\psi_s(k),\\
-C_{1,1}^{(i)}&=\frac{\sqrt{4\pi}}{\mathbf{K}_2}\frac{g(k)k}{K}C_+^{(i)}+\sum_{s=1}^{\infty}c_{+s}^{(i)}\psi_s(k),\\
C_{1,-1}^{(i)}&=\frac{\sqrt{4\pi}}{\mathbf{K}_2}\frac{g(k)k}{K}C_-^{(i)}+\sum_{s=1}^{\infty}c_{-s}^{(i)}\psi_s(k),\\
C_{1,0}^{(i)}&=\frac{\sqrt{4\pi}}{\mathbf{K}_2}\frac{g(k)k}{K}C_0^{(i)}+\sum_{s=1}^{\infty}c_{0s}^{(i)}\psi_s(k)
\end{aligned}\right\} \tag{1. 19}$$

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(11) loc. cit. (6)

where the functions $\psi_s(k)$ $(s=1,\ 2,\ 3\ \ldots)$ make a complete system of normalized orthogonal functions with $\dfrac{\sqrt{4\pi}}{\mathbf{K}_2}\dfrac{g(k)k}{K}\left(\mathbf{K}_2^2=4\pi\int\dfrac{g(k)^2k^2}{K^2}dk\right)$ as its first member. $A_0^{(i)}$, $C_+^{(i)}$, $C_-^{(i)}$, $C_0^{(i)}$, $a_{0s}^{(i)}$, $c_{+s}^{(i)}$, $c_{-s}^{(i)}$ and $c_{0s}^{(i)}$ satisfy then the following commutation relations:

$$\left.\begin{array}{l}[A_0^{(i)},\ A_0^{(i)*}]=[C_+^{(i)},\ C_+^{(i)*}]=[C_-^{(i)},\ C_-^{(i)*}]=[C_0^{(i)}\ \ C_0^{(i)*}]=1,\\ [a_{0s}^{(i)},\ a_{0s'}^{(i)*}]=[c_{+s}^{(i)},\ c_{+s'}^{(i)*}]=[c_{-s}^{(i)},\ c_{-s'}^{(i)*}]=[c_{0s}^{(i)},\ c_{0s'}^{(i)*}]=\delta_{ss'}\\ \text{other commutation relations}=0\end{array}\right\}. \tag{1.19}$$

We denote Schrödinger equation (1.17) divided by $\hbar c$ as

$$(H-W_0')\Psi=0. \tag{1.20}$$

Substituting (1.18) in H or (1.9), it follows;

$$\begin{aligned}H=&\sum_{i=1}^{2}\left\{A_0^{(i)*}A_0^{(i)}+C_+^{(i)*}C_+^{(i)}+C_-^{(i)*}C_-^{(i)}+C_0^{(i)*}C_0^{(i)}\right\}\frac{\mathbf{K}_1^3}{\mathbf{K}_2^2}\\ &+\frac{l}{\sqrt{2}}\frac{\mathbf{K}_1^3}{\mathbf{K}_2}\sum_{i=1}^{2}(A_0^{(i)*}-A_0^{(i)})\tau_i+l'\frac{\mathbf{K}_1^3}{\mathbf{K}_2}\sum_{i=1}^{2}\Big\{(C_+^{(i)}+C_-^{(i)*})\sigma+(C_-^{(i)}+C_+^{(i)*})\sigma^*\\ &+\frac{1}{\sqrt{2}}(C_0^{(i)*}+C_0^{(i)})\sigma_z\Big\}\tau_i+\sum_{i=1}^{2}\sum_s H_s'\left(\frac{1}{\mathbf{K}_2}A_0^{(i)*}-\frac{l}{\sqrt{2}}\tau_i\right)a_{0s}^{(i)}\\ &+\sum_{i=1}^{2}\sum_s H_s'^*\left(\frac{1}{\mathbf{K}}A_0^{(i)}+\frac{l}{\sqrt{2}}\tau_i\right)a_{0s}^{(i)*}+\sum_{i=1}^{2}\sum_s H_s'\Big\{\left(\frac{C_+^{(i)*}}{\mathbf{K}}+l'\sigma\tau_i\right)c_{+s}^{(i)}\\ &+\left(\frac{C_-^{(i)*}}{\mathbf{K}_2}+l'\sigma\tau_i\right)c_{-s}^{(i)}+\left(\frac{C_0^{(i)*}}{\mathbf{K}_2}+l'\frac{\sigma_z}{\sqrt{2}}\tau_i\right)c_{0s}^{(i)}\Big\}+\sum_{i=1}^{2}\sum_s H_s'\Big\{\Big(\frac{C_+^{(i)}}{\mathbf{K}_2}\\ &+l'\sigma^*\tau_i\Big)c_{+s}^{(i)*}+\left(\frac{C_-^{(i)}}{\mathbf{K}_2}+l'\sigma^*\tau_i\right)c_{-s}^{(i)*}+\left(\frac{C_0^{(i)}}{\mathbf{K}_2}+l'\frac{\sigma_z}{\sqrt{2}}\tau_i\right)c_{0s}^{(i)*}\Big\}\\ &+\sum_{i=1}^{2}\sum_{s,s'}H_{ss'}(a_{0s}^{(i)*}a_{0s'}^{(i)}+c_{+s}^{(i)*}c_{+s'}^{(i)}+c_{-s}^{(i)*}c_{-s'}^{(i)}+c_{0s}^{(i)*}c_{0s'}^{(i)})\end{aligned} \tag{1.21}$$

where

$$\sigma=\frac{\sigma_x+i\sigma_y}{2},\qquad \sigma^*=\frac{\sigma_x-i\sigma_y}{2},\qquad \mathbf{K}_1^3=4\pi\int\frac{g(k)^2k^2}{K}dk,$$

$$H_s'=\sqrt{4\pi}\int g(k)k\psi_s(k)dk,\qquad H_s'^*=\sqrt{4\pi}\int g(k)k\psi_s^*(k)dk,$$

$$H_{ss'}=\int K\psi_s^*(k)\psi_s(k)dk.$$

We introduce the operators $n_a^{\pm}$, $n_{as}^{\pm}$; $n_+^{\pm}$, $n_{+s}^{\pm}$; $n_-^{\pm}$, $n_{-s}^{\pm}$, $n_0^{\pm}$, and $n_{0s}^{\pm}$ by

$$\left.\begin{aligned}
&A_0^{(1)*}A_0^{(1)}=n_a^+, \quad A_0^{(2)*}A_0^{(2)}=n_a^-, \quad a_{0s}^{(1)*}a_{as}^{(1)}=n_{0s}^+, \quad a_{0s}^{(2)*}a_{0s}^{(2)}=n_{as}^-\\
&C_+^{(1)*}C_+^{(1)}=n_+^+, \quad C_+^{(2)*}C_+^{(2)}=n_+^-, \quad c_{+s}^{(1)*}c_{+s}^{(1)}=n_{+s}^+, \quad c_{+s}^{(2)*}c_{+s}^{(2)}=n_{+s}^-\\
&C_-^{(1)*}C_-^{(1)}=n_-^+, \quad C_-^{(2)*}C_-^{(2)}=n_-^-, \quad c_{-s}^{(1)*}c_{-s}^{(1)}=n_{-s}^+, \quad c_{-s}^{(2)*}c_{-r}^{(2)}=n_{-s}^-\\
&C_0^{(1)*}C_0^{(1)}=n_0^+, \quad C_0^{(2)*}C_0^{(2)}=n_0^-, \quad c_{0s}^{(1)*}c_{0s}^{(1)}=n_{0s}^+, \quad c_{0s}^{(2)*}c_{0s}^{(2)}=n_{0s}^-
\end{aligned}\right\} \qquad (1.22)$$

whose eigenvalues are zero and positive integers and physically mean respectively the " Besetzungszahlen " of longitudinal and transverse mesons, the superscript signs that of meson charges. If we neglect the fourth and the succeeding terms in (1. 21), the eigenfunction considered as the function of these " Besetzungszahlen " for the lowest state becomes approximately:

$$\Psi=\Phi(n_a^+,\ n_a^-,\ n_+^+,\ n_+^-,\ n_-^+,\ n_-^-,\ n_0^+,\ n_0^-)\prod_{s=1}^{\infty}\delta_{n_s,0} \qquad (1.23)$$

where the four rowed function

$$\Phi=\begin{cases}\varphi_1(n_a^+,\ n_a^-,\ n_+^+,\ n_+^-,\ n_-^+,\ n_-^-,\ n_0^+,\ n_0^-)\\ \varphi_2(\qquad\qquad\qquad '' \qquad\qquad\qquad)\\ \psi_1(\qquad\qquad\qquad '' \qquad\qquad\qquad)\\ \psi_2(\qquad\qquad\qquad '' \qquad\qquad\qquad)\end{cases} \qquad (1.24)$$

satisfies the following equation

$$\begin{aligned}\sum_{i=1}^{2}\Big\{&A_0^{(i)*}A_0^{(i)}+C_+^{(i)*}C_+^{(i)}+C_-^{(i)*}C_-^{(i)}+C_0^{(i)*}C_0^{(i)}+\frac{V_1}{\sqrt{2}}(A_0^{(i)*}-A_0^{(i)})\tau_i\\ &+V_2[(C_+^{(i)}+C_-^{(i)*})\sigma+(C_-^{(i)}+C_+^{(i)*})\sigma^*+\frac{1}{\sqrt{2}}(C_0^{(i)*}+C_0^{(i)})\sigma_z]\tau_i\Big\}\Phi=0\end{aligned} \qquad (1.25)$$

with

$$V_1=l\mathbf{K}_2, \qquad V_2=l'\mathbf{K}_2.$$

Now it is convenient to pass from the Schrödinger equation (1. 25) to the differential equations, by the aid of the transformation theory in the quantum mechanics; we introduce namely the operators $\xi_1^{(i)}$, $\xi_2^{(i)}$, $\xi_3^{(i)}$, $\xi_4^{(i)}$, $p_1^{(i)}$, $p_2^{(i)}$, $p_3^{(i)}$ and $p_4^{(i)}$ connected with $A_0^{(i)*}$, $A_0^{(i)}$, $C_+^{(i)*}$, $C_+^{(i)}$. $C_-^{(i)*}$, $C_-^{(i)}$, $C_0^{(i)*}$ and $C_0^{(i)}$ by

$$A_0^{(i)*}-A_0^{(i)}=\sqrt{2}\,\xi_4^{(i)}, \quad C_+^{(i)}+C_-^{(i)*}=\xi_1^{(i)}-i\xi_2^{(i)},$$

$$C_+^{(i)}-C_-^{(i)*}=p_1^{(i)}-ip_2^{(i)}, \quad C_0^{(i)*}+C_0^{(i)}=\sqrt{2}\,\xi_3^{(i)},$$

$$A_0^{(i)*}+A_0^{(i)}=i\sqrt{2}\,p_4^{(i)},\quad C_+^{(i)*}+C_-^{(i)}=\xi_1^{(i)}+i\xi_2^{(i)},$$
$$C_+^{(i)*}-C_-^{(i)}=p_1^{(i)}+ip_2^{(i)},\quad C_0^{(i)*}-C_0^{(i)}=i\sqrt{2}\,p_3^{(i)}. \qquad (1.26)$$

Since these operators $\xi^{(i)}$ and $p^{(i)}$ are obviously Hermitic and satisfy the following commutation relations:

$$[\xi_k^{(i)},\ p_l^{(j)}]=i\delta_{ij}\delta_{kl},\ [\xi_k^{(i)},\ \xi_l^{(j)}]=[p_k^{(i)},\ p_l^{(j)}]=[\xi_k^{(i)},\ \xi_l^{(i)}]=[p_k^{(i)},\ p_l^{(i)}]=0$$
$$k,\ l=1,\ 2,\ 3,\ 4 \qquad (1.27)$$

$p_k^{(i)}$ can be identified with the operator $\dfrac{1}{i}\dfrac{\partial}{\partial\xi_k^{(i)}}$, if we use $\xi_k^{(i)}$ representation for Φ. Thus (1.25) takes the desired form:

$$\sum_{i=1}^{2}\left\{-\frac{1}{2}\sum_{k=1}^{4}\frac{\partial^2}{\partial\xi_k^{(i)2}}+\frac{1}{2}\sum_{k=1}^{4}\xi_k^{(i)2}-4+V_1\xi_4^{(i)}\tau_i+V\,(\xi_1^{(i)}\sigma_x+\xi_2^{(i)}\sigma_y+\xi_3^{(i)}\sigma_z)\tau_i\right.$$
$$\left.-W_0\right\}\Phi(\xi_1^{(1)},\ \xi_2^{(1)},\ \xi_3^{(1)},\ \xi_4^{(1)},\ \xi_1^{(2)},\ \xi_2^{(2)},\ \xi_3^{(2)},\ \xi_4^{(2)})=0 \qquad (1.28)$$

As far as we concern only the transverse mesons, the wave functions $\Phi_t(\xi_1^{(1)},\ \xi_2^{(1)},\ \xi_3^{(1)},\ \xi_1^{(2)},\ \xi_2^{(2)},\ \xi_3^{(2)})$ for the transverse mesons satisfy the following differential equations:

$$\sum_{i=1}^{2}\left\{-\frac{1}{2}\Delta_{\xi^{(i)}}+\frac{1}{2}\vec{\xi}^{(i)2}-W_t-3+V_2(\vec{\xi}^{(i)},\vec{\sigma})\tau_i\right\}\Phi_t(\vec{\xi}^{(1)},\vec{\xi}^{(2)})=0 \qquad (1.29)$$

where

$$\vec{\xi}^{(i)}=(\xi_1^{(i)},\ \xi_2^{(i)},\ \xi_3^{(i)}).$$

The solution of (1.29) corresponding to the lowest state of the proton interacting with transverse mesons can be shown to be

$$\Phi_t=\begin{vmatrix}1\\0\\0\\0\end{vmatrix}f_1(X,\ Y)+\begin{vmatrix}0\\0\\\xi_3^{(1)}\mp i\xi_3^{(2)}\\\xi_1^{(1)}+i\xi_1^{(2)}+i(\xi_3^{(1)}+i\xi_2^{(2)})\end{vmatrix}f_2(X,\ Y)$$
$$+\begin{vmatrix}[\vec{\xi}^{(1)},\ \vec{\xi}^{(2)}]_3\\ [\vec{\xi}^{(1)},\ \vec{\xi}^{(2)}]_1+i[\vec{\xi}^{(1)},\ \vec{\xi}^{(2)}]_2\\0\\0\end{vmatrix}f_3(X,\ Y)$$

$$-i\begin{vmatrix} 0 \\ 0 \\ [\vec{\xi}^{(1)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_3+i[\vec{\xi}^{(2)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_3 \\ [\vec{\xi}^{(1)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_1+i[\vec{\xi}^{(2)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_1+i\{[\vec{\xi}^{(1)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_2+i[\vec{\xi}^{(2)},[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]]_2\} \end{vmatrix}$$

$$\times f_4(X,Y)=\begin{vmatrix} \varphi_1 \\ \varphi_2 \\ \psi_1 \\ \psi_2 \end{vmatrix} \qquad (1.30)$$

where

$$X=\vec{\xi}^{(1)2}+\vec{\xi}^{(2)2},\quad Y=[\vec{\xi}^{(1)},\vec{\xi}^{(2)}]^2 \qquad (1.31)$$

and φ_1, φ_2; ψ_1, ψ_2 express the states of proton and of neutron with spin ↑ or ↓ respectively. Since by writing Φ_t in this form the 'angular parts" have seperated, substitution of (1. 30) in (1. 29) gives the differential equations for the " radial parts " f_s' including only X and Y, where the variables X and Y can be regarded as the generalized radius vectors in the 6-dimensional space of $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$. The differential equations for f_s' are solved approximately for the case $V_2\gg 1$ as follows:

$$f_1=\frac{R_1(R_1^2-R_2^2)}{4}\,\frac{1}{2}\mathring{f}(R_1,R_2),\qquad f_2=\frac{R_1^2-R_2^2}{4}\,\frac{1}{2}\mathring{f}(R_1,R_2),$$

$$f_3=R_1\frac{1}{2}\mathring{f}(R_1,R_2),\qquad f_4=\frac{1}{2}\mathring{f}(R_1,R_2),$$

$$\mathring{f}(R_1,R_2)=e^{-\frac{\varepsilon_1^2}{4}}H_n(\varepsilon_1)e^{-\frac{\varepsilon_2^2}{4}}L_m\left(\frac{\varepsilon_2^2}{2}\right) \qquad (1.32)$$

where

$$R_1=\sqrt{X+2\sqrt{Y}},\quad R_2=\sqrt{X-2\sqrt{Y}},\quad R_1=2V_1+\varepsilon_1,\quad R_2=\varepsilon_2.$$

$H_n(x)$ and $L_m(x)$ being Hermite's and Laguerre's polynomials respectively. The required eigenvalue W_t for the lowest state $n=m=0$ is

$$W_t=\frac{3}{2}-V_2^2+\frac{5}{16V_2^2}.$$

The self energy of the proton due to the transverse c-components or charged vector meson field is accordingly, for strong coupling $V_2 = l'\mathbf{K}_2 \gg 1$,

$$\mathbf{K}_{0t}^{V_2 \gg 1} = -l'^2\mathbf{K}_1^3 - \frac{3}{2}\frac{\mathbf{K}_1^3}{\mathbf{K}_2^2} + \frac{5}{16}\frac{\mathbf{K}_1^3}{l'^2\mathbf{K}_2^4}. \tag{1. 33}$$

This is compared with that given by the perturbational method, i. e.

$$\mathbf{K}_{0t}^{V_2 \ll 1} = -3l'^2\mathbf{K}_1^3 \tag{1. 34}$$

and is also as large, here 3 times by a factor as that for strong coupling.

§ 2

As we have obtained the wave functions representing the states of mesons bounded to a nucleon for strong coupling, we now proceed to calculate the anomalous magnetic moment of the proton, regarding the interaction with electromagnetic field as a perturbation. The perturbation energy is, taking that part refered to the transverse components of charged vector mesons which only contributes to the magnetic moment of the proton, given by

$$H_v' = \frac{1}{2\pi\varkappa^2}\frac{e}{\hbar c}\int(\vec{A}, [\vec{U}^{(2)}, \operatorname{rot}\vec{U}^{(1)}] - [\vec{U}^{(1)}, \operatorname{rot}\vec{U}^{(2)}])\,d\tau$$
$$+ \frac{2eg_2}{\varkappa^2\hbar c}\int(\vec{A}, [\vec{U}^{(2)}, \vec{\sigma}\tau_1] - [\vec{U}^{(1)}, \vec{\sigma}\tau_2])\,\delta(\vec{r})\,d\tau \tag{2. 1}$$

where $\vec{A}$ denotes the vector potential of electromagnetic field and in our case of uniform magnetic field H_{0z} becomes

$$A_x = -\frac{1}{2}H_{0z}y, \quad A_y = \frac{1}{2}H_{0z}x, \quad A_z = 0. \tag{2. 2}$$

Substituting (1. 3) and (2. 2) in (2, 1), the second term in (2. 1) vanishes, while the first reduces to

$$H_v' = \frac{1}{4}\hbar c\frac{e}{\hbar c}H_{0z}\iint\frac{\delta(k-k')}{\sqrt{KK'}}dk\,dk'2\Big\{\Big(C_{1,1}^{(1)}(k) - C_{1,-1}^{(1)}{}^*(k)\Big)$$
$$\times\Big(C_{1,-1}^{(2)}(k') - C_{1,1}^{(2)*}(k')\Big) - \Big(C_{1,-1}^{(1)}(k) - C_{1,1}^{(1)*}(k)\Big)\Big(C_{1,1}^{(2)}(k') - C_{1,-1}^{(2)*}(k')\Big)\Big\} \tag{2. 3}$$

which is, by using (1. 18) and (1. 26) simplified as follows:

$$H_v' = \hbar c \frac{e}{\hbar c} H_{0_2} \frac{\mathrm{K}_3}{\mathrm{K}_2^2} [\vec{\xi}^{(1)}, \vec{\xi}^{(2)}]. \qquad (2.\ 4)$$

It is proved that only P mesons with the angular momentum parallel to z-axis have magnetic energy and thus produce the effect. It is noted that (2. 4) also, as in § 1, resembles the magnetic energy in the case of charged pseudoscalar mesons. In fact, for the latter we have

$$H_{ps}' = \frac{e}{2\pi\hbar c} \int (\vec{A},\ Q_2 \operatorname{grad} Q_1 - Q_1 \operatorname{grad} Q_2) d\tau. \qquad (2.\ 5)$$

By (1. 13) and the formulas as same as (1. 18) and (1. 26), (2. 5) can be expressed in the form:

$$H_{ps}' = \hbar c \frac{e}{\hbar c} H_{0_2} \frac{\mathrm{K}_3}{\mathrm{K}_2^2} [\vec{\xi}^{(1)}, \vec{\xi}^{(2)}]_3 \qquad (2.\ 6)$$

Treating H_v' (2. 4) as a perturbation, the self energy of the proton is corrected in the first order by the amount

$$H' = \int \Phi_t^* H_v' \Phi_t d\vec{\xi}^{(1)} d\vec{\xi}^{(2)} \Big/ \int \Phi_t^* \Phi_t d\vec{\xi}^{(1)} d\vec{\xi}^{(2)} \qquad (2.\ 7)$$

Inserting in this the wave functions (1. 32) with $n=m=0$, we obtain as the anomalous magnetic moment of the proton

$$\mu_v^{V_2 \gg 1} = \hbar c \frac{e}{\hbar c} \frac{\mathrm{K}_3}{\mathrm{K}_2^2} \times 0.13\, V_2^2 = 0.07 \times \frac{4}{3\pi} \frac{e}{\hbar c} \frac{g_2^2}{\varkappa_2} \int \frac{k^4}{(k^2+\varkappa^2)^1} dk. \qquad (2.\ 8)$$

while in the case of weak coupling one has obtained by means of the perturbation theory⁽¹²⁾

$$\mu_v^{V_2 \ll 1} = \frac{4}{3\pi} \frac{e}{\hbar c} \frac{g_2^2}{\varkappa^2} \int \frac{k^4}{(k_2+\varkappa_2)_2} dk. \qquad (2.\ 9)$$

In the pseudoscalar meson theory, as seen from (2. 4) and (2. 6), these values are reduced to just a half for strong as well as weak coupling, except the difference of the interaction constants. If we evaluate the integral in (2. 9)

$$\mu_v^{V_2 \ll 1} = \frac{8}{3\pi} \frac{g_2^2}{\hbar c} \frac{M}{m_0} \mu_0 \left\{ \frac{\mathrm{K}}{\varkappa} + \frac{1}{2} \frac{\mathrm{K}\varkappa}{\mathrm{K}^2+\varkappa^2} - \frac{3}{2} \tan^{-1} \frac{\mathrm{K}}{\varkappa} \right\} \qquad (2.\ 10)$$

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(12) loc. cit. (7).

where μ_0 is a nuclear magneton $e\hbar/2Mc$, M being the rest mass of a proton. If the cutting-off momentum $\mathbf{K}$ and the interaction constant $g_2^2/\hbar c$ are assumed to be 10κ[13] and 0.3 respectively, the former of which is determind, comparing the theoretical scattering cross section of vector mesons by nucleons with cosmic ray experiments and the latter is given by the magnitude of the range of nuclear forces, the anomalous magnetic moment given by (2. 10) amounts to

$$\mu_v^{V_2 \ll 1} = 20\mu_0\,, \tag{2. 11}$$

which is evidently too large compared with the experimental value. On the contrary, for strong coupling we have for it the value $1.4\mu_0$, which is more or less smaller than the observed value. Thus it seems to show that we should deal with the case of not so strong but somewhat moderate coupling. This is rather accentuated for the case of the pseudoscalar meson theory, as the anomalous magnetic moment is, in this case, as half times as that of vector meson theory. Although the above values $\mathbf{K}=10\kappa$ and $g_2^2/\hbar c=0.3$ are somewhat problematic, since they were determined by means of the longitudinal vector meson theory, and we should, in order to be self-consistent, determine these constants in the frame of the transverse vector meson theory. It may be still expected that the so determined constants will not be very much different from the values assumed above, thus not altering essentially the above conclusion.

In conclusion, the author wishes to express his cordial thanks to Dr. Y. Nishina, Prof. M. Yamada and Prof. S. Tomonaga for their kind interest and encouragement and to Assistant Prof. K. Nakabayashi for his valuable discussions. It is noted that the cost of this research has been defrayed from the Scientific Research Expenditure of the Department of Education.

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[13] loc. cit. (3).

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On the Meson Pair Theory of Nuclear Forces, I.

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§ 1. Introduction and Summary.

As yet no decisive conclusion has been drawn about the spin of the meson. The experiment⁽¹⁾ and the theory⁽²⁾ of the cosmic ray shower by the hard component, however, seem to indicate the value 0 or $\hbar/2$. While in the single meson theory of nuclear forces the spin of the meson must be taken to be integral, in the pair theory we have two possibilities of taking the spin to be half integral as well as integral. Many authors have studied the pair theory with mesons of spin 0⁽³⁾ or $\hbar/2$⁽⁴⁾. But unfortunately the latter theory in the current form, involving charged mesons only, can not explain the phenomena of the β–decay.

To remove this defect we need only to replace one of the charged mesons of the pair by a neutral one and to assume the following processes to exist

$$\left.\begin{array}{l} P \rightleftarrows N + Y^{+} + Y^{0}, \\ N \rightleftarrows P + Y^{-} + Y_{0}, \end{array}\right\} \qquad (1)$$

where Y^{+}, Y^{-} and Y^{0} denote a positive, negative, and neutral meson respectively.

In this paper, the scalar meson pair theory of nuclear forces modified in this direction is presented. We begin with the discussion of the possible form of the interaction between the nuclear particles and the meson fields leading to processes of the type (1) (§ 2), and the potential between the neutron and the proton is deduced (§ 3). Then the interaction constant is dertermined from the cross section for the scattering of mesons by nuclear particles (§ 4). Finally, we show that it is possible in the modified pair theory to construct a theory of the β-decay, which leads to the interaction of Fermi type between the nuclear particles and the light particles (§ 5).

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(1) Schein and Gill, Rev. Mod. Phys. **11**, (1939) 267.

(2) Christy and Kusaka, Phys. Rev. **59**, (1941) 414.

(3) Wentzel, Zs. f. Phys. **118**, (1941) 277; Wentzel, Helv. Phys. Acta **16**, (1942) 111.

(4) Marschak, Phys. Rev. **57**, (1940) 1101; Marschak and Weisskopf, ibid. **59**, (1941) 130; Critchfield and Lamb, ibid. **58**, (1940) 46; Critchfield, ibid. **59**, (1941) 48; Weinberg, ibid. (1941) 776; Jauch, Helv. Phys. Acta **14**, (1941) 465; **15**, (1942) 175; **16**, (1942) 221; Miyazima, Sci. Pap. I.P.C.R. **39**, (1941) 28; Teketani, Kagaku **11**, (1941) 523; Ozaki, Proc. Phys.-Math. Soc. Japan **16**, (1942) 209.

§ 2. Interaction of Nuclear Particles with the Meson Fields.

We introduce the mesons with spin 0 and describe the charged meson field by a scalar complex wave function U and its conjugate complex $\tilde{U}$, and the neutral meson field by a real wave function V. In vacuum, they satisfy the Klein-Goldon equations

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\varkappa^2\right)F=0. \tag{2}$$

where F stands for one of the three wave functions mentioned above and $\varkappa$ is related to the mass m_U of the meson by $\varkappa=m_U c/\hbar$. We consider here for simplicity the masses of the charged and the neutral meson to be equal. Now these equations can be derived from the Lagrangian(5)

$$\left.\begin{aligned}\overline{L}_U&=\iiint L_U\,dv,\\ \text{with}\qquad L_U&=\left\{\frac{1}{c^2}\frac{\partial\tilde{U}}{\partial t}\frac{\partial U}{\partial t}-\operatorname{grad}\tilde{U}\cdot\operatorname{grad}U-\varkappa^2\tilde{U}U\right\}\\ &\quad+\frac{1}{2}\left\{\frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2-(\operatorname{grad}V)^2-\varkappa^2V^2\right\}.\end{aligned}\right\} \tag{3}$$

We now proceed to the formulation of the most general interaction of the nuclear particles with the meson fields leading to processes of the type (1). Assuming the Dirac wave function for the nuclear particle, we form the covariants

$$\left.\begin{aligned}&\text{Scalar :}\qquad && W=\tilde{\Psi}(\beta\tau_{NP})\Psi,\\ &\text{Vector :}\qquad && \vec{M}=\tilde{\Psi}(\vec{\alpha}\tau_{NP})\Psi;\quad M_0=\tilde{\Psi}\tau_{NP}\Psi,\end{aligned}\right\} \tag{4}$$

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(5) Pauli, Huitieme Conseil de Physique Solvay Kap. II. § 1.

where Ψ and $\widetilde{\Psi}$ are the wave functions for the nuclear particle, which have eight components. $\vec{\alpha}$ and β are the Dirac matrices and τ_{NP} is the operator changing the nuclear particle from the proton to the neutron state.

We assume that the interaction satisfies the following conditions :

(i) The interaction term in the total Lagrangian involves only the wave function of the nuclear particle without its derivatives.

(ii) It contains only the meson wave function and their first derivatives.

(iii) It must not be of the higher degree than the first with respect to the derivatives.

Then the general interaction which is relativistically invariant will be a linear combination of the following two terms only :

$$L_g=-gW\widetilde{U}V+\text{conj. comp.}, \tag{5}$$

$$\left.\begin{aligned} L_f=&-f_1\left\{\vec{M}\cdot\widetilde{U}\,\text{grad}\,V+\frac{1}{c}M_0\widetilde{U}\frac{\partial V}{\partial t}\right\}-f_2\Big\{(\vec{M}\cdot\text{grad}\,\widetilde{U})V \\ &+\frac{1}{c}M_0\frac{\partial\widetilde{U}}{\partial t}V\Big\}+\text{conj. comp.}, \end{aligned}\right\} \tag{6}$$

where f_1, f_2 and g are arbitrary real constants with the dimensions of area and length respectively. The justification of the assumptions (i) and (ii) have already been discussed by Kemmer⁽⁶⁾. The assumption (iii) is necessary in order that we can quantize the field in the usual procedure.

The Lagrangian for the total system consisting of the nuclear particles and the meson fields can now be written in the form

$$\overline{L}=\iiint(L_U+L_g+L_f+L_n)dv, \tag{7}$$

where L_n is the Lagrangian for the free nuclear particle. This is of course well known, as we assume the Dirac equation for the nuclear particle, and we need not put it down explicitly.

Defining the canonically conjugate F^+ of any field variable F in the usual way,

$$F^+=\frac{\delta\overline{L}}{\delta\dot{F}},$$

we obtain

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(6) Kemmer, Proc. Roy, Soc. 166, (1938) 127.

$$\left.\begin{aligned} U^{+}&=\frac{1}{c^2}\frac{\partial\widetilde{U}}{\partial t}-\frac{f_2}{c}\widetilde{M}_0V,\\ V^{+}&=\frac{1}{c^2}\frac{\partial V}{\partial t}-\frac{f_1}{c}(M_0\widetilde{U}+\widetilde{M}_0U),\\ \Psi_{\rho}^{+}&=i\hbar\widetilde{\Psi}_{\rho}\quad\text{for }\rho=1,\ 2,\ldots\ldots 8.\end{aligned}\right\}\qquad(8)$$

The total Hamiltonian can thus be obtained by the usual procedure. We can express it as follows:

$$\overline{H}=\overline{H}_U+\overline{H}_i+\overline{H}_n\,,\qquad(9)$$

Here $\overline{H}_U$ denotes the energy of the meson fields in vacuum and $\overline{H}_n$ the energy of the free nuclear particles.

They are given by

$$\left.\begin{aligned}\overline{H}_U=\iiint\Big[&\Big\{c^2\widetilde{U}^{+}U^{+}+\operatorname{grad}\widetilde{U}\cdot\operatorname{grad}U+\varkappa^2\widetilde{U}U\Big\}\\ &+\frac{1}{2}\Big\{c^2(V^{+})^2+(\operatorname{grad}V)^2+\varkappa^2V^2\Big\}\Big]dv\,,\end{aligned}\right\}\qquad 10)$$

$$\overline{H}_n=\iiint\widetilde{\Psi}\left\{c\vec{\alpha}\cdot\vec{p}+\beta\left(\frac{1+\tau_3}{2}M_nc^2+\frac{1-\tau_3}{2}M_pc^2\right)\right\}\Psi\,dv\,,\qquad(11)$$

where τ_3 is the istopic spin matrix and $\vec{p}=-i\hbar\operatorname{grad}$. M_n and M_p are the masses of the neutron and the proton respectively.

$\overline{H}_i$ is the energy of interaction between the nuclear particles and the meson fields and is of the form

$$\left.\begin{aligned}\overline{H}_i=\overline{H}_g+\overline{H}_f+\overline{H}_{f2}&=\iiint H_g\,dv+\iiint H_f\,dv+\iiint H_{f2}\,dv\,,\\ H_g&=gW\widetilde{U}V+\text{conj. comp.},\\ H_f&=f_1\Big\{cM_0\widetilde{U}V^{+}+\vec{M}\cdot\widetilde{U}\operatorname{grad}V\Big\}\\ &\quad+f_2\Big\{cM_0U^{+}V+(\vec{M}\cdot\operatorname{grad}\widetilde{U})V\Big\}+\text{conj. comp.},\\ H_{f2}&=\frac{1}{2}f_1^2(M_0U+M_0U)^2+\frac{1}{2}f_2^2(M_0\widetilde{M}_0+\widetilde{M}_0M_0)^2V^2.\end{aligned}\right\}\qquad(12)$$

Having found the Hamiltonian, quantization is simple. We postulate the commutation rules

$$\left.\begin{array}{c}[U(\vec{x},t),\ U^{+}(\vec{x}',t)]=[\widetilde{U}(\vec{x},t),\ \widetilde{U}^{+}(\vec{x}',t)]=[V(\vec{x},t),\ V^{+}(\vec{x}',t)]=i\hbar\,\delta(\vec{x}-\vec{x}'),\\ \Psi_{\rho}(\vec{x},t)\widetilde{\Psi}_{\rho'}(\vec{x}',t)+\widetilde{\Psi}_{\rho'}(\vec{x}',t)\Psi_{\rho}(\vec{x},t)=\delta_{\rho\rho'}\,\delta(\vec{x}-\vec{x}'),\\ \Psi_{\rho}(\vec{x},t)\Psi_{\rho'}(\vec{x}',t)+\Psi_{\rho'}(\vec{x}',t)\Psi_{\rho}(\vec{x},t)=\widetilde{\Psi}_{\rho}(\vec{x},t)\widetilde{\Psi}_{\rho'}(\vec{x}',t)+\widetilde{\Psi}_{\rho'}(\vec{x}',t)\widetilde{\Psi}_{\rho}(\vec{x},t)=0.\end{array}\right\} \tag{13}$$

All other pairs are comnutable. Their time derivatives are then calculated by the usual rule

$$i\hbar\dot{F}=[F,\ \overline{H}].$$

Now each of the terms in the expression for $\overline{H}_{f2}$ gives rise to an infinitely large interaction between the nuclear particles and the meson fields. This is most easily shown by passing to the description of the state of the nuclear particles in their configuration space. In this respresentation we may put[7]

$$M_0=\widetilde{\Psi}\tau_{NP}\Psi=\sum_{s}\tau_{NP}^{(s)}\,\delta(\vec{x}-\vec{X}^{(s)}),$$

where $\tau_{NP}^{(s)}$ and $\vec{X}^{(s)}$ are the corresponding matrix and coordinates which reffer to the s-th nuclear particle. Then the first term in $\overline{H}_{f2}$, for instance, takes the form

$$\left.\begin{array}{l}\overline{H}_{f_1^2}=\dfrac{1}{2}f_1^2\displaystyle\sum_{s\neq t}\Big\{\widetilde{U}(\vec{X}^{(s)})U(\vec{X}^{(s)})\tau_{NP}^{(s)}\tau_{PN}^{(t)}+U(\vec{X}^{(s)})\widetilde{U}(\vec{X}^{(s)})\tau_{PN}^{(s)}\tau_{NP}^{(t)}\\ \quad+\widetilde{U}(\vec{X}^{(s)})\widetilde{U}(\vec{X}^{(s)})\tau_{NP}^{(s)}\tau_{NP}^{(t)}+U(\vec{X}^{(s)})U(\vec{X}^{(s)})\tau_{PN}^{(s)}\tau_{PN}^{(t)}\Big\}\,\delta(\vec{X}^{(s)}-\vec{X}^{(t)})\\ \qquad\qquad+\dfrac{1}{2}f_1^2\displaystyle\sum_{s}\widetilde{U}(\vec{X}^{(s)})U(\vec{X}^{(s)})I\delta(\vec{X}^{(s)}-\vec{X}^{(s)}),\end{array}\right\} \tag{14}$$

where τ_{PN} is the reverse operator to τ_{NP} and I the unit operator in the istopic spin space. The terms in the first and the second line in (14) give, as a second order process, an interaction between different particles of the form of δ-functions. The last term resembles the interaction given by Wentzel[8] in his pair theory

$$\overline{H}_{\lambda}=\lambda\sum_{s}\widetilde{U}(\vec{X}^{(s)})U(\vec{X}^{(s)}), \tag{15}$$

but gives an infinitely large interaction on account of the factor $\delta(\vec{X}^{(s)}-\vec{X}^{(s)})$.

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(7) Fock, Zs. f. Phys. 75, (1932) 622.

(8) Wentzel, loc. cit.

Similar considerations hold for the second term in H_{f2}.

In order to have a finite interaction, we should therefore take $f_1=f_2=0$. In other words, *the interaction term in the pair theory should not contain the derivatives of the meson wave functions.* This assumption will be necessary irrespective of the spin of the meson assumed. In accordance with this new assumption we take the simple interaction $\overline{H}_t=\overline{H}_g$ in the following.

For the purposes of calculation we go to the momentum space of the mesons. We assume all the functions to be periodic in some very large volume V, and change the variables describing the meson fields by the Fourier transformation⁽⁹⁾

$$\left.\begin{aligned}
U&=\frac{\hbar c}{\sqrt{2V}}\sum_{\vec{k}}\frac{1}{\sqrt{E_k}}\left\{a_{+1}(\vec{k})e^{i\vec{k}\cdot\vec{x}}+a^*_{-1}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right\},\\
V&=\frac{\hbar c}{\sqrt{2V}}\sum_{\vec{k}}\frac{1}{\sqrt{E_k}}\left\{a_{0}(\vec{k})e^{i\vec{k}\cdot\vec{x}}+a^*_{0}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right\},\\
U^+&=\frac{i}{c\sqrt{2V}}\sum_{\vec{k}}\sqrt{E_k}\left\{a^*_{+1}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}-a_{-1}(\vec{k})e^{i\vec{k}\cdot\vec{x}}\right\},\\
V^+&=\frac{i}{c\sqrt{2V}}\sum_{\vec{k}}\sqrt{E_k}\left\{a^*_{0}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}-a_{0}(\vec{k})e^{i\vec{k}\cdot\vec{x}}\right\},
\end{aligned}\right\}\tag{16}$$

where $E_k=\hbar c\sqrt{\kappa^2+|\vec{k}|^2}$. The new variables $a_\varepsilon(\vec{k})$ and $a^*_\varepsilon(\vec{k})$ ($\varepsilon=\pm1$ or 0) satisfy the commutation rules

$$[a_\varepsilon(\vec{k}),\ a^*_{\varepsilon'}(\vec{k}')]=\delta_{\varepsilon\varepsilon'}\delta_{\vec{k}\vec{k}'},\tag{17}$$

all other pairs commuting.

We can show that $N^\varepsilon(\vec{k})=a^*_\varepsilon(\vec{k})a_\varepsilon(\vec{k})$ can be interpreted as the number of the mesons with charge εe in the state of energy E_k and momentum $\hbar\vec{k}$. In fact the energy of the meson fields in vacuum expressed in the new variables has the form

$$\overline{H}_U=\sum_{\varepsilon}\sum_{\vec{k}}E_k\left\{N^\varepsilon(\vec{k})+\frac{1}{2}\right\}.\tag{18}$$

Inserting (16) in the expression for H_g as given in (12) we have

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(9) Pauli, loc. cit.

$$\bar{H}_g = g\frac{\hbar^2 c^2}{2V}\sum_s \sum_{\vec{k}} \sum_{\vec{k}'} \frac{\beta^{(s)}}{\sqrt{E_k E_{k'}}}\Big[\Big\{\overset{*}{a}_{+1}(k)\overset{*}{a}_0(\vec{k}')\tau^{(s)}_{NP}$$

$$+\overset{*}{a}_{-1}(\vec{k})\overset{*}{a}_0(\vec{k}')\tau^{(s)}_{PN}\Big\}e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}^{(s)}}$$

$$+\Big\{\overset{*}{a}_{+1}(\vec{k})a_0(\vec{k}')\tau^{(s)}_{NP}+\overset{*}{a}_{-1}(\vec{k})a_0(\vec{k}')\tau^{(s)}_{PN}\Big\}e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}^{(s)}}+\text{conj. comp.}\Big]. \tag{19}$$

This interaction will be considered as a small perturbation in the actual calculations. It involves the operators which cause the transitions (1) and those which lead to scattering processes besides.

§ 3. The Interaction between Neutrons and Protons.

The neutron and the proton are coupled with each other by the mutual emission and reabsorption of a pair of a charged and a neutral meson. The potential energy between them at distance r apart is given by the second order perturbation formula. After some calculations we obtain for the potential in the non-relativistic approximation

$$\left.\begin{aligned} &V_{NP}(r)=(\tau^{(1)}_{NP}\tau^{(2)}_{PN}+\tau^{(1)}_{PN}\tau^{(2)}_{NP})\,K(r)\,,\\ \text{with}\qquad &K(r)=g^2\cdot\frac{\hbar c\varkappa}{32\pi^2}\cdot\frac{H_1^{(1)}(2i\varkappa r)}{r^2}\,,\end{aligned}\right\} \tag{20}$$

where $H_1^{(1)}(2i\varkappa r)$ is the Hankel function of the first order.

This potential gives an exchange force of Heisenberg type between the neutron and the proton. It decreases very rapidly at large distances and behaves as r^{-3} at small distances, as the Hankel function has the asymptotic forms

$$H_1^{(1)}(2i\varkappa r)=\begin{cases}-\dfrac{e^{-2\varkappa r}}{\sqrt{\pi\varkappa r}} & \text{for}\quad 2\varkappa r\gg 1\,.\\[2ex] -\dfrac{1}{\pi\varkappa r} & \text{for}\quad 2\varkappa r\ll 1\,.\end{cases} \tag{21}$$

It is therefore necessary to cut off the potential at small distance comparable to $(2\varkappa)^{-1}$, which is the range of force. If we take for it the value 2×10^{-13} cm, we have

$$m_U \sim 100\,\text{m}\quad(\text{m}\,;\ \text{electron mass}). \tag{22}$$

The potential given above is repulsive in the ^3S state and attrative in the ^1S state of the deuteron, which is in contradition with experience. Now Wentzel⁽¹⁰⁾ has assumed the interaction (15) between the nuclear particles and the meson field and derived the static potential between two nuclear particles, which holds for any value of the interaction constant. For small interaction, it reduces to the expression $K(r)$ as given in (20) with $g=\lambda$ and is thus always attractive. We can not, therefore, depress the triplet state below the singlet by combining the interactions of the types (15) and (19) only.

Further, with the value of g determined from the cross section for the scattering of mesons by nuclear particles (see § 4), the absolute value of the potential $V_{NP}(r)$ becomes too small at small distances, i. e. about two thousand times smaller than that given by Bethe⁽¹¹⁾ on the neutral theory. This is due to the circumstances that the scattering process is merely of the first order in our case, whereas the neutron-proton interaction is given by the second order pertubation. The situation is therefore more or less characteristic of the pair theory. But it may be remedied to a certain extent by choosing the cut-off distance smaller.

Whether or not these defects can be removed by introducing the mesons with spin $\hbar/2$ will be discussed in the next paper.

§ 4. Scattering of Mesons by Nuclear Particles.

Next, we shall apply the theory to the problem of scattering of mesons by nuclear particles, which is of interest in connexion with the collision of mesons with atomic nuclei. There are four cases to be considered which are represented schematically below.

(i) $Y^+ + N \rightarrow P + Y^0$,

(ii) $Y^0 + N \rightarrow P + Y^-$,

(iii) $Y^0 + P \rightarrow N + Y^+$,

(iv) $Y^- + P \rightarrow N + Y^0$.

It is easily seen that the cross section is the same for all these processes. We calculate it in the system in which the nuclear particle is initially at rest. We shall give only the result here. Averaging and summing over the two possible spin states of the nuclear particle in its initial and final states respectively, we get for the differential cross section that the mesons

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(10) Wentzel, loc. cit.

(11) Bethe, Phys. Rev. 57, (1940) 390,

are scattered into the solid angle $d\Omega$ in the direction θ, φ, the expression

$$d\sigma=\frac{g^2}{32\pi^2}\left\{\frac{Mc}{Mc+p_0(1-\cos\theta)}\right\}\left\{1+\frac{p_0^2(1-\cos\theta)}{Mc\{Mc+p_0(1-\cos\theta)\}}\right\}d\Omega, \tag{23}$$

where p_0 is the momentum of the incident meson and M the mass of the nuclear particle. In the above calculation we have neglected $(m_v/M)^2$ compared with 1. Integrating (23) with respect to angles, we get a simple expression for the total cross section

$$\sigma=\frac{g^2}{8\pi}\cdot\left(\frac{1+\gamma}{1+2\gamma}\right)^2, \tag{24}$$

where $$\gamma=p_0/Mc\,.$$

Although the value of σ is not known experimentally, yet the cross section for the scattering in which the charge of the meson is not altered has been determined experimentally by Wilson(12). If we may assume that the two cross sections are the same(13), we can use the experimental value for the latter to determine the constant g. Thus we find for the dimensionless constant $\varkappa g$

$$(\varkappa g)^2\sim\frac{1}{30}, \tag{25}$$

with the value of g as given by (22) and (25), the cross section (24) is always small, being of the order $10^{-28}\,\mathrm{cm}^2$.

Moreover, it is evident that there is no possibility of absorption of a meson by the nuclear particle on the pair theory. All these circumstances seem to favour the interpretation of the large penetrating power of the cosmic ray meson.

§ 5. Theory of β-decay.

In order to explain the phenomena of the β-decay according to our theory, we propose the existence of the following processes between the mesons and the light particles

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(12) Wilson, Proc. Roy. Soc. **174**, (1940) 73.

(13) Indeed, if we assume that the interaction between the nuclear particles and the meson fields consists of two terms given by (15) and (19), and take $\lambda=g$, the two cross sections become the same.

$$\left.\begin{array}{l} \nu \rightleftarrows e^- + Y^+ + Y^0, \\ e^- \rightleftarrows \nu + Y^- + Y^0, \end{array}\right\} \tag{26}$$

where ν and e^- denote a neutrino and an electron respectively

These processes are analogous to those given in (1). Thus the same reasoning as given in § 2 leads us to choose the following expression for the interaction which produces these processes:

$$\left.\begin{array}{l} \overline{H}_{g'} = g' \iiint w\widetilde{U} V dv + \text{conj. comp.}, \\ w = \widetilde{\psi}\beta\varphi. \end{array}\right\} \tag{27}$$

where

ψ and φ are the wave functions of the electron and the neutrino respectively. The constant g' has the dimension of length.

Then the β-disintegration can take place as the result of the combination of the processes (1) and (26). Namely, a neutron can change into a proton with the emission of a pair of a charged and a neutral meson. This pair of mesons is absorbed subsequently by a neutrino in the negative energy state, which is thereby transformed into an electron of positive energy state. Similar processes lead to a radioactive transformation in which a positron is emitted.

The matrix element H_β for the β-decay process can easily be calculated by the usual perturbation theory. Now we can neglect the energy of disintegration compared with $2m_U c^2$, as the former is smaller than the latter for all the known β-disintegrations. Then the calculation of H_β is just the same as that of the neutron-proton interaction and gives

$$H_\beta = gg' \frac{\hbar c \varkappa}{32\pi^2} \int \cdots \int \frac{H_1^{(1)}(2i\varkappa r)}{r^2} \widetilde{\Psi}_p(\vec{x}_1)\beta\Phi_n(\vec{x}_1)\widetilde{\psi}_s(\vec{x}_2)\beta\varphi_o(\vec{x}_2)\,dv_1 dv_2\,, \tag{28}$$

Here $r = |\vec{x}_1 - \vec{x}_2|$. Φ_n and Ψ_p are the eigenfunctions of the nuclear particle before and after emission. φ_o and ψ_s are the similar quantities for the light particle. Since the wave length of the emitted light particles are large compared with $(2\varkappa)^{-1}$, we can further regard φ_o and ψ_s as constant in the region of length $(2\varkappa)^{-1}$. Thus we obtain finally

$$H_\beta = gg'\,C \frac{\hbar c \varkappa}{8\pi} \iiint \widetilde{\Psi}_p(\vec{x})\beta\Phi_n(\vec{x})\widetilde{\psi}_s(\vec{x})\beta\varphi_o(\vec{x})\,dv, \tag{29}$$

$$C = \int \frac{H_1^{(1)}(2i\varkappa r)}{r^2} r^2 dr\,. \tag{30}$$

Thus our theory gives the interaction energy of Fermi type between the nuclear particles and the light particles. Here it is also necessary to cut off the potential at small distances as in the case of the neutron-proton interaction (see § 3). To obtain an estimate of the magnitude of C, we may use as the integrand in (30) the asymptotic form of $H_1^{(1)}$ for large r given in (21) and integrate it from 0 to ∞, instead of cutting it off at small distance comparable to $(2\varkappa)^{-1}$. Then we obtain

$$C \sim \frac{1}{\varkappa}. \tag{31}$$

With the values of g and C as given in (25) and (31), we can determine the constant g'. Thus we find that the ratio g'/g should be taken to be about one thousand times greater than the corresponding quantity in the single scalar meson theory⁽¹⁴⁾. In view of the situation mentioned at the end of § 3, this can be easily understood.

§ 6. Concluding Remarks.

With the interaction as given by (27), it seems that a charged meson may disappear even in the free space by emitting a positive or a negative electron and neutral particles, i. e. a neutral meson and a neutrino or an antineutrino, according as the charge of the meson is positive or negative. However, these processes are not allowed so long as we take the masses of the charged and the neutral meson to be equal. Because the laws of conservation of energy and momentum can not be satisfied simultaneously in these processes. In order to give a finite lifetime of the charged meson, therefore, it will be necessary to assume the mass of the neutral meson to be smaller than that of the charged one.

Furthermore, as mentioned in § 3, it would be necessary to introduce the meson with spin $\hbar/2$ in order to give the correct interaction between the neutron and the proton. In the next paper we will develop the theory in this direction.

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(14) Yukawa; Proc. Phys.-Math. Soc. Japan **17**, (1935) 48.

In conclusion, the writer wishes to express his cordial thanks to Professor H. Yukawa and to Professor M. Kobayasi for the interest taken in this work and the valuable suggeations given.

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On the Effect of the Field Reactions on the Interaction of Mesotrons and Nuclear Particles, I.*

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Introduction.

In studying the interactions of mesotrons with nuclear particles the perturbation theory has been first applied without being earnestly doubted whether this process will give convergent results in these problems. In the quantum electrodynamics which deals with the interaction between electromagnetic field and electrons this process has given the results which agree very well with experimental evidences so that one can scarcely doubt the applicability of this process in this case. Still, there are several reasons for believing that the perturbation theory would not be applicable to the case of mesotron theory.

The interaction between mesotron field and nuclear particles differs in many respects from that between electromagnetic field and electrons. In the first place, the former interaction is much intensiver than the latter: the coupling constant g specifying the former interaction is, at least, about ten times so large as the constant e which specifies the latter. In the second place, the potential of the mesotronic self-field of a nuclear particle has a singularity of the form $1/r^3$ or $\delta(r)$ which is a stronger singularity** than that of the electrostatic self-field of an electron where the potential is of the form of $1/r$. This fact means that the interaction between nuclear particle with its mesotronic self-field is much more singular than that between electron and its electric self-field. It is, thus, rather doubtfull whether the perturbation theory does converge in case of our mesotron theory, since this procedure presupposes the weakness of the coupling.

Under such circumstances several authors have attempted to solve the problem involving the interaction of the nuclear particle with the mesotron field by some other ways without using the rather problematic perturbation theory. The aim of the present paper is to give a review of treatments of this kind and to make clear the underlying ideas and discuss the results obtained by these treatments.

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* Translated from the paper appeared in Bull. I.P.C.R., 22 (1943), 781.

** In the scalar mesotron theory the singularity is exceptionally of the form of $1/r$.

That the perturbation theory can not be applicable to our problem is implied by a simple exaple. One calculates the scattering cross-section of a mesotron on colliding with a nuclear particle. Supposing thereby that the nuclear particle is infinitely heavy so that in this collision its recoil can be neglected, one finds, according to the ordinary perturbation theory, that the cross-section increases indefinitely with increasing energy of the colliding mesotron. It is, however, unreasonable that in such a problem the cross-section exceeds the square of the wave length of the incident particle.

Physically means the application of the perturbation theory the neglect of the field reaction. The fact that the scattering cross-section exceeds the square of the wave length of the colliding particle means that in such a case this reaction is by no means small. What prevents the cross-section from increasing so unreasonablly must be just this field reaction.

The reaction of a field on a particle can be classified into two distinct kinds, each of which has physically as well as mathematically different properties. The first one of these reactions is caused by the following phenomena: Around the particle there exist the so-called self-field. But the field has inertia against its change. When, now, the state of motion of the particle changes, also its self-field must change, so that it exhibits the inertia and an inertial reaction acts on the particle. Since this reaction is, in the first approximation, proportional to the acceleration of the particle, this reaction results in an appearent change of its mass. The other one of the field reactions is the effect of damping which is caused in the following manner: When the particle is in a state of non-uniform motion, it disturbs the surrounding field. Then waves propagate from this center, so that energy will be brought away by these waves, because under the natural boundary condition only outgoing waves can exist at infinity. This phenomena results thus in the dissipation of energy of the particle and, consequently, an irreversible damping force acts on the particle.

As is well known, the inertial reaction of the field is infinitely large in the existing theory, so that one is forced to touch on the fundamental defect of the quantum field theory whenever one will deal with this reaction. Thus the existing theory is quite powerless to give account of such phenomena in which the first term of the perturbation theory does not suffice and the field reaction takes an essential part. One can get rid of this difficulty only by a very unsatisfactory procedure of "cut-off."*

Although there exists at present no unumbiguous way of introducing the cut-off process in general cases, it seems fairly certain that this process corresponds in some way to giving a finite size to the particle interacting with the field. In the simplified problem in which the particle is considered as infinitely heavy, this process of giving the finite size is introduced in a comparatively unumbiguous way. So we shall in this paper deal with this simplified problem, and ask in what way the nuclear particle with infinite mass and finite size will interact with the mesotron field.

When the problem is simplified in this way, it can be solved also when the coupling is not weak. Thus one can see in what manner the field will react on the particle. When the coupling of the field and the particle is very strong, this can be done by solving the problem classically, since we may expect that in this case the field generated by the particle is so large that the quantum-theoretical fluctuation can be neglected throughout. Indeed, it is seen by the quantum-theoretical treatment of Wentzel, which is a powerfull method applicable to the case of strong coupling, that the classical theory gives a satisfactory approximation in case of strong coupling. Wentzel has namely treated the problem quantum-theoretically expanding the solution in inverse powers of the coupling constant, whereas in the ordinary perturbation theory one finds the solution in powers of the coupling constant.

If the interaction is not strong enough and neither classical nor Wentzel's treatment is allowed, the problem can be solved approximately by Ritz's procedure. This method gives a reasonable results which connect the results in the extreme cases of strong and weak couplings: Wentzel's results on one hand and the results of the perturbation theory on the other hand.

In the present paper we shall deal with the problem successively by classical, by Wentzel's and by Ritz's methods. We shall show first that the results of the classical treatment do really agree with the quantum-theoretical results of Wentzel when the coupling is strong. We shall then treat the problem by Ritz's procedure. These treatments give rise to several results which differ substantially from those obtained by the perturbation theory. These differences represent just the effects of the field reaction. We shall discuss in the last paragraph the physical consequences of these treatments.

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* There is another method of avoiding the divergence difficulties of the theory. Namely, one subtracts the infinite terms of the field reaction appearing in the course of calculations. This procedure is based on the hypothesis that the inertial reaction has no physical meanings at all. A group of works has been appeared on the basis of this "subtraction" procedure.

§ 1. Classical treatment.(1) (2) (3)

Although in case of strong coupling a quantum-theoretical treatment by means of Wentzel's method is possible, it is still of interest to treat the problem first classically, since in this way we can obtain a clear picture of the behavior of field and particle when they interact with each other.

A:—Equations of motion.

We suppose throughout the present paper that the mesotron field is described by a four-potential, thus we adopt the vector theory of the mesotron field. We assume, for simplicity, that of two Yukawa's constants g_1 and g_2 the latter vanishes. This assumption means that only longitudinal mesotrons interact with the nuclear particle. As to the charge of the mesotrons, we use at will sometimes the so-called symmetrical theory and sometimes the charged theory aiming at the simplicity of the treatment.

We now consider that the nuclear particle lies at the coordinate origin. Let the mesotron field be described by means of the (properly normalized) amplitudes of the normal vibrations. Then, adopting the symmetrical theory, the Hamiltonian function of our system is

$$\begin{aligned} H = & \int K\Big\{a^*(\mathfrak{k})a(\mathfrak{k})+b^*(\mathfrak{k})b(\mathfrak{k})+c^*(\mathfrak{k})c(\mathfrak{k})\Big\}d\mathfrak{k} \\ & -\frac{l}{2}\int \frac{kF(k)}{\sqrt{K}}\Big[\Big\{a(\mathfrak{k})-b^*(\mathfrak{k})\Big\}(\tau_1+i\tau_2)+\Big\{a^*(\mathfrak{k})-b(\mathfrak{k})\Big\}(\tau_1-i\tau_2) \\ & +\sqrt{2}\Big\{c(\mathfrak{k})+c^*(\mathfrak{k})\Big\}\tau_3\Big]d\mathfrak{k}, \end{aligned} \tag{1·1}$$

where $a(\mathfrak{k})$, $b(\mathfrak{k})$ and $c(\mathfrak{k})$ are the amplitudes of the normal vibrations with the momentum $\mathfrak{k}$. They satisfy the commutation relations

$$\left.\begin{array}{c} [a(\mathfrak{k}),\ a^*(\mathfrak{k}')]=[b(\mathfrak{k}),\ b^*(\mathfrak{k}')]=[c(\mathfrak{k}),\ c^*(\mathfrak{k}')]=\delta(\mathfrak{k}-\mathfrak{k}') \\ \text{all other commutators}=0. \end{array}\right\} \tag{1·2}$$

If we introduce n^+, n^- and n^0 by means of

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(1) W Heisenberg: Z. Phys., **113** (1939), 61.

(2) J. R. Oppenheimer and J. Schwinger: P. R **60** (1941), 150.

(3) T. Miyazima and S. Tomonaga: Sc. pap. I.P.C.R., **40** (1943), 274

$$n^+(\mathfrak{k})=a^*(\mathfrak{k})\,a(\mathfrak{k}),\ n^-(\mathfrak{k})=b^*(\mathfrak{k})\,b(\mathfrak{k}),\ n^0(\mathfrak{k})=c^*(\mathfrak{k})\,c(\mathfrak{k}), \tag{1·3}$$

then $n^+(\mathfrak{k})d\mathfrak{k}$, $n^-(\mathfrak{k})d\mathfrak{k}$ and $n^0(\mathfrak{k})d\mathfrak{k}$ mean respectively the numbers of the positive, the negative and the neutral mesotrons whose momenta lie in the small volume $d\mathfrak{k}$ at the point $\mathfrak{k}$ in the momentum space. In (1·1) it is supposed that the natural unit system is used in which $\hbar$ and c are unity. Then momenta as well as energies have the dimension of 1/length. The quantity K means the energy of the mesotron having the momentum $\mathfrak{k}$ and is given by

$$K=\sqrt{k^2+\varkappa^2}\qquad k=|\mathfrak{k}|, \tag{1·4}$$

if $\varkappa$ denotes the rest energy of the mesotron.

The three quantities τ_1, τ_2 and τ_3 in (1·1) are the three components of the isotopic spin τ which satisfy the following relations

$$[\tau_1,\ \tau_2]=2i\tau_3,\ [\tau_2,\ \tau_3]=2i\tau_1,\ [\tau_3,\ \tau_1]=2i\tau_2 \tag{1·5}$$
$$\tau_1^2=\tau_2^2=\tau_3^2=1.$$

Further, the constant l in (1·1) is the coupling constant having the dimension of the length. It is related to Yukawa's g_1 by means of

$$l=\frac{1}{2\pi}\,\frac{g_1}{\sqrt{\hbar c}}\,\frac{1}{\varkappa}. \tag{1·6}$$

The function $F(k)$ is the so-called "cutting off" factor introduced in order to avoid the divergence of the theory. The introduction of this factor corresponds to the assumption of the finite size of the nuclear particle. When the radius of the particle is denoted by R and its reciprocal by $\boldsymbol{K}$:

$$R=\frac{1}{\boldsymbol{K}}, \tag{1·7}$$

then $F(k)$ has the property

$$F(k)=\begin{cases}0 & \text{for } k\gtrsim\boldsymbol{K}\\ 1 & \text{for } k\lesssim\boldsymbol{K}.\end{cases} \tag{1·8}$$

We call $\boldsymbol{K}$ "cut-off momentum" hereafter.

We can now obtain from the Hamiltonian (1·1) and the commutation relations (1·2) and (1·5) the equations of motion for the amplitudes a, b and c and for the isotopic spin components τ_1, τ_2 and τ_3 in the ordinary manner. For a, b and c they are

$$\left.\begin{aligned}\dot{a}(\mathfrak{k})&=-i\left\{Ka(\mathfrak{k})-\frac{l}{2}g(k)(\tau_1-i\tau_2)\right\}\\ \dot{b}(\mathfrak{k})&=-i\left\{Kb(\mathfrak{k})+\frac{l}{2}g(k)(\tau_1+i\tau_2)\right\}\\ \dot{c}(\mathfrak{k})&=-i\left\{Kc(\mathfrak{k})-\frac{i}{\sqrt{2}}g(k)\tau_3\right\}\end{aligned}\right\} \qquad (1\cdot 9)$$

and for τ_1, τ_2 and τ_3

$$\left.\begin{aligned}\dot{\tau}_1&=l\int g(k)\left[\frac{1}{i}\left\{a(\mathfrak{k})-a^*(\mathfrak{k})+b(\mathfrak{k})-b^*(\mathfrak{k})\right\}\tau_3+\sqrt{2}\left\{c(\mathfrak{k})+c^*(\mathfrak{k})\right\}\tau_2\right]d\mathfrak{k}\\ \dot{\tau}_2&=l\int g(k)\left[\left\{a(\mathfrak{k})+a^*(\mathfrak{k})-b(\mathfrak{k})-b^*(\mathfrak{k})\right\}\tau_3-\sqrt{2}\left\{c(\mathfrak{k})+c^*(\mathfrak{k})\right\}\tau_1\right]d\mathfrak{k}\\ \dot{\tau}_3&=il\int g(k)\left[\left\{a(\mathfrak{k})-b^*(\mathfrak{k})\right\}(\tau_1+i\tau_2)-\left\{a^*(\mathfrak{k})-b(\mathfrak{k})\right\}(\tau_1-i\tau_2)\right]d\mathfrak{k},\end{aligned}\right\} \qquad (1\cdot 10)$$

where $g(k)$ is the abreviation

$$g(k)=\frac{kF(k)}{\sqrt{K}}. \qquad (1\cdot 11)$$

Having thus obtained the equations of motion, we can now treat the problem classically. In this way our problem is considerablly simplified, and at the same time we can obtain a simple picture of the interaction under consideration, which also serves much the clear understanding of the quantum-theoretical treatment of the problem. We consider thus that a, b and c are no more q-numbers but ordinary c-numbers, and that the isotopic spin τ is an ordinary unit vector in the 123–space. Although the problem is in this way very much simplified, it is still hardly possible to obtain the general solutions of the equations of motion. But, fortunately, the discussions of the problems concerning the self-field around the nuclear particle, or the problem involving the scattering of a mesotron by the nuclear particle can be carried out without knowing such general solutions, for these purposes it suffices only to know rather simple special solutions of the equations of motion. We shall thus proceed to these simple problems.

B :—Self-field of a nuclear particle.

Since our main purpose is to obtain a picture of the behavior of field and nuclear particle interacting with each other, it is preferable to use real amplitudes instead of complex a, b and c. So we introduce real variables

$\xi_1(\mathfrak{k})$, $\xi_2(\mathfrak{k})$ and $\xi_3(\mathfrak{k})$ and their canonically conjugate momenta $p_1(\mathfrak{k})$, $p_2(\mathfrak{k})$ and $p_3(\mathfrak{k})$ by means of

$$\left.\begin{aligned}
\xi_1(\mathfrak{k})-i\xi_2(\mathfrak{k})&=\frac{1}{\sqrt{K}}\left\{a(\mathfrak{k})-b^*(\mathfrak{k})\right\}\\
\xi_1(\mathfrak{k})+i\xi_2(\mathfrak{k})&=\frac{1}{\sqrt{K}}\left\{a^*(\mathfrak{k})-b(\mathfrak{k})\right\}\\
p_1(\mathfrak{k})-ip_2(\mathfrak{k})&=-i\sqrt{K}\left\{a(\mathfrak{k})+b^*(\mathfrak{k})\right\}\\
p_1(\mathfrak{k})+ip_2(\mathfrak{k})&=i\sqrt{K}\left\{a^*(\mathfrak{k})+b(\mathfrak{k})\right\}\\
\xi_3(\mathfrak{k})&=\frac{1}{\sqrt{2K}}\left\{c(\mathfrak{k})+c^*(\mathfrak{k})\right\}\\
p_3(\mathfrak{k})&=-i\sqrt{K/2}\left\{c(\mathfrak{k})-c^*(\mathfrak{k})\right\}.
\end{aligned}\right\}\qquad(1\cdot12)$$

It is easily seen that these amplitudes and momenta satisfy the canonical commutation relations

$$\left.\begin{aligned}
[\xi_j(\mathfrak{k}),\ p_{j'}(\mathfrak{k}')]&=i\delta_{jj'}\delta(\mathfrak{k}-\mathfrak{k}')\\
[\xi_j(\mathfrak{k}),\ \xi_{j'}(\mathfrak{k}')]&=0\\
[p_j(\mathfrak{k}),\ p_{j'}(\mathfrak{k})]&=0,\quad j,j'=1,2,\text{ or }3.
\end{aligned}\right\}\qquad(1\cdot13)$$

In terms of these variables the Hamiltonian function is expressed as

$$H=\frac{1}{2}\int\Big[\left\{p_1(\mathfrak{k})^2+K^2\xi_1(\mathfrak{k})^2\right\}+\left\{p_2(\mathfrak{k})^2+K^2\xi_2(\mathfrak{k})^2\right\}+\left\{p_3(\mathfrak{k})^2+K^2\xi_3(\mathfrak{k})^2\right\}\Big]d\mathfrak{k}$$
$$-l\int g(k)\sqrt{K}\left\{\xi_1(\mathfrak{k})\tau_1+\xi_2(\mathfrak{k})\tau_2+\xi_3(\mathfrak{k})\tau_3\right\}d\mathfrak{k}.\qquad(1\cdot14)$$

It is seen that this Hamiltonian is invariant with respect to the rotations in the 123–space, so that the "angular momentum" in this space is a constant of motion. The components of this angular momentum are defined by

$$\left.\begin{aligned}
J_1&=\int\left\{\xi_2(\mathfrak{k})p_3(\mathfrak{k})-\xi_3(\mathfrak{k})p_2(\mathfrak{k})\right\}d\mathfrak{k}+\frac{1}{2}\tau_1\\
J_2&=\int\left\{\xi_3(\mathfrak{k})p_1(\mathfrak{k})-\xi_1(\mathfrak{k})p_3(\mathfrak{k})\right\}d\mathfrak{k}+\frac{1}{2}\tau_2\\
J_3&=\int\left\{\xi_1(\mathfrak{k})p_2(\mathfrak{k})-\xi_2(\mathfrak{k})p_1(\mathfrak{k})\right\}d\mathfrak{k}+\frac{1}{2}\tau_3.
\end{aligned}\right\}\qquad(1\cdot15)$$

If we reprsent a vector in the 123–space by a thick letter (a vector in the xyz–space has been represented by a German letter), (1 · 15) can be written as

$$\mathbf{J}=\mathbf{L}+\frac{1}{2}\boldsymbol{\tau},\quad \mathbf{L}=\int\Big[\boldsymbol{\xi}(\mathfrak{k}),\ \mathbf{p}(\mathfrak{k})\Big]d\mathfrak{k},\qquad (1\cdot16)$$

where the first term $\mathbf{L}$ gives the "mesotronic angular momentum" and the second term $\frac{1}{2}\boldsymbol{\tau}$ the "spin angular momentum."

The equations of motion for $\boldsymbol{\xi}(\mathfrak{k})$, $\mathbf{p}(\mathfrak{k})$ and $\boldsymbol{\tau}$ are obtained either by transforming (1 · 9) and (1 · 10) or directly from H of (1 · 14):

$$\left.\begin{aligned}\dot{\boldsymbol{\xi}}(\mathfrak{k})&=\mathbf{p}(\mathfrak{k})\\ -\dot{\mathbf{p}}(\mathfrak{k})&=K^2\boldsymbol{\xi}(\mathfrak{k})-lg(k)\sqrt{K}\,\boldsymbol{\tau}\\ \dot{\boldsymbol{\tau}}&=-2l\int g(k)\sqrt{K}\Big[\boldsymbol{\xi}(\mathfrak{k}),\ \boldsymbol{\tau}\Big]d\mathfrak{k}\,.\end{aligned}\right\}\qquad (1\cdot17)$$

Now, in order to find the self-field of the nuclear particle, we calculate first the state of the lowest energy. Classically this state corresponds to the state of equilibrium of the system in which all quantities do not change with the time. Thus we obtain from (1·17) by putting $\dot{\boldsymbol{\xi}}(\mathfrak{k})=0$, $\mathbf{p}(\mathfrak{k})=0$, $\dot{\boldsymbol{\tau}}=0$:

$$\left.\begin{aligned}\mathbf{p}(\mathfrak{k})&=0\\ \boldsymbol{\xi}(\mathfrak{k})&=\frac{lg(k)}{K\sqrt{K}}\boldsymbol{\tau}\end{aligned}\right\}\qquad (1\cdot18)$$

in which $\boldsymbol{\tau}$ is a constant unit vector independent of time. One sees that in this state the "mesotronic" angular momentum vanishes so that we have

$$\mathbf{J}=\frac{1}{2}\boldsymbol{\tau}.\qquad (1\cdot19)$$

Of course, $\mathbf{J}$ can take any direction in the purely classical theory. In the quantum theory, however, it must be quantized. Then the vector $\boldsymbol{\tau}$ can only be either parallel or antiparallel to the 3-rd axis, so that J_3 is necessarily 1/2 or $-1/2$. Since the third component of the angular momentum has the physical meaning that, added by 1/2, it gives the charge of the system this result means that these two equilibrium states correspond just to the proton state and the neutron state respectively.

We must now obtain more general states of our system in order to find the possible self-fields of the nuclear particle. This is necessary because, in the first place, a self-field of a particle does not necessarily correspond to the state of equilibrium, but there are possibilities of several excited self-filds in which the time derivatives of various quantities do not vanish. Indeed, every state of motion in which the field does not extend to infinity but it exists only in the neighbourhood of the particle gives a self-field. In the second place, it is because the states of the lowest angular momentum do not exactly correspond to the state of no rotation if we will correct our result so as to fit better the quantum theory.

To obtain such general states of motion we put

$$\left.\begin{array}{lll} \tau_1=\sin\theta\cos\omega t, & \tau_2=\sin\theta\sin\omega t, & \tau_3=\cos\theta \\ \xi_1=A_1\cos\omega t, & \xi_2=A_2\sin\omega t, & \xi_3=A_3 \\ p_1=B_1\sin\omega t, & p_2=B_2\cos\omega t, & p_3=B_3 \end{array}\right\} \tag{1·20}$$

and determine the unknown constants A_1, A_2, A_3, B_1, B_2, B_3 and ω so that (1·20) satisfies the equations of motion. We consider, namely, the state in which the vector τ makes a precessional motion with the angular velocity ω around the third axis making the angle θ with it, and at the same time, the field rotates with the same angular velocity.

Introducing (1·20) into (1·17) and equating the coefficients of $\sin\omega t$ and $\cos\omega t$ on both sides, we obtain

$$\left.\begin{array}{c} \xi_1=\dfrac{lg(k)\sqrt{K}}{K^2-\omega^2}\sin\theta\cos\omega t, \quad \xi_2=\dfrac{lg(k)\sqrt{K}}{K^2-\omega^2}\sin\theta\sin\omega t, \\ \xi_3=\dfrac{lg(k)}{K\sqrt{K}}\cos\theta, \\ \omega=2l^2\cos\theta\displaystyle\int g(k)^2\left\{\frac{K}{K^2-\omega^2}-\frac{1}{K}\right\}d\mathfrak{k}, \end{array}\right\} \tag{1·21}$$

or in terms of the complex amplitudes

$$\left.\begin{array}{l} a(\mathfrak{k})=\dfrac{1}{K-\omega}\,\dfrac{l}{2}g(k)\sin\theta\, e^{-i\omega t} \\ b(\mathfrak{k})=-\dfrac{1}{K+\omega}\,\dfrac{l}{2}g(k)\sin\theta\, e^{i\omega t} \\ c(\mathfrak{k})=\dfrac{1}{K}\,\dfrac{l}{\sqrt{2}}g(k)\cos\theta\,. \end{array}\right\} \tag{1·22}$$

We now obtain the energy of the system in this state. Inserting (1·22) into (1·1), or (1·21) into (1·14) we find then

$$\begin{aligned} H=&\frac{l^2}{2}\int g(k)^2 K\left\{\frac{K^2+\omega^2}{(K^2-\omega^2)^2}\sin^2\theta+\frac{1}{K^2}\cos^2\theta\right\}d\mathfrak{k} \\ &-l^2\int g(k)^2\left\{\frac{K}{K^2-\omega^2}\sin^2\theta+\frac{1}{K}\cos^2\theta\right\}d\mathfrak{k}\,, \end{aligned} \tag{1·23}$$

which is to be interpreted as the self-energy of the nuclear particle in this state. Also the angular momentum of the system is found by substituting (1·21) into (1·15):

$$\left.\begin{aligned} J_1&=-\omega l^2\sin\theta\cos\theta\cos\omega t\int\frac{g(k)^2}{K(K^2-\omega^2)}d\mathfrak{k}+\frac{1}{2}\sin\theta\cos\omega t \\ J_2&=-\omega l^2\sin\theta\cos\theta\sin\omega t\int\frac{g(k)^2}{K(K^2-\omega^2)}d\mathfrak{k}+\frac{1}{2}\sin\theta\sin\omega t \\ J_3&=\omega l^2\sin^2\theta\int\frac{g(k)^2}{(K^2-\omega^2)^2}d\mathfrak{k}+\frac{1}{2}\cos\theta\,. \end{aligned}\right\} \tag{1·24}$$

As mentioned above, J_3 means the charge number (viz. charge–1/2), which is usually denoted by m, so we have

$$m=\omega l^2\sin^2\theta\int\frac{g(k)^2}{(K^2-\omega^2)^2}d\mathfrak{k}+\frac{1}{2}\cos\theta\,. \tag{1·25}$$

In this expression the first term is to be interpreted as the charge belonging to the mesotronic field around the nuclear particle, and the second term as the charge proper to the nuclear particle. The charge obserbed is the sum of these two.

Of course, in the purely classical theory m can take arbitrary values, but if we introduce quantization, it can take only half odd values (positive as well as negative). In the quantum theory m is thus a quantum number which we shall call " charge quantum number."

From the relations (1·21), (1·23) and (1·25) we see that our motion is perfectly determined if we specify the value of m. Thus H and ω can be represented as functions of m. It is rather tedious to calculate these functions in general cases, but this is meaningless since the classical treatment is valid only when the coupling is strong. In such a case we find $\omega\ll\varkappa\ll K$ and this enables us to calculate the required functions expanding various quantities in powers of $\omega/\varkappa$ and ω/K. This simplifies the calculations considerablly and we obtain

$$\left.\begin{aligned} \cos\theta&=\frac{1}{2m}, \quad \omega=\frac{m}{l^2 \boldsymbol{K}_3} \\ H&=-\frac{l^2}{2}\boldsymbol{K}_1^3+\frac{1}{2l^2\boldsymbol{K}_3}\left(m^2-\frac{1}{4}\right) \end{aligned}\right\} \qquad (1\cdot 26)$$

where the quantity $\boldsymbol{K}_n$ (n : integer) is defined by

$$\boldsymbol{K}_n=\left\{\int \frac{g(k)^2}{K^n}dk\right\}^{1/4-n}. \qquad (1\cdot 27)$$

These $\boldsymbol{K}_n$ have the dimension of 1/length, and, when n is smaller than 4, they are of the order of magnitude of $\boldsymbol{K}$, which we assume to be very large as compared with $\varkappa$.

In this case of strong coupling it is seen from (1·26) that ω is very small. Thus the speed of the precession of the system is very slow. In order to see the detailed nature of this motion, we calculate now the angular momentum in this case of strong coupling. For $\omega \ll \boldsymbol{K}$ we obtain from (1·24) approximately

$$\left.\begin{aligned} L_1&=-\frac{1}{2}\sin\theta\cos\omega t \\ L_2&=-\frac{1}{2}\sin\theta\sin\omega t \\ L_3&=m\sin^2\theta\,. \end{aligned}\right\} \qquad (1\cdot 28)$$

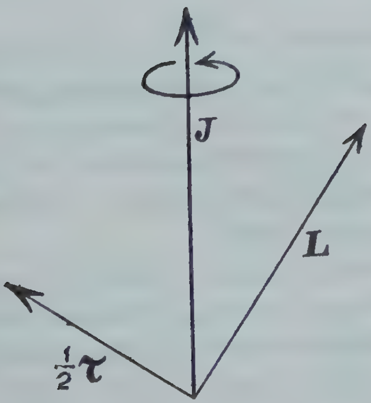

Fig. 1. Three vectors $\frac{1}{2}\tau$, **L** and **J**

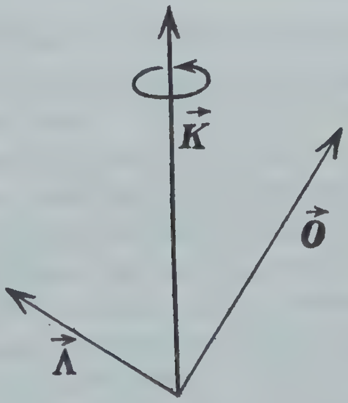

Fig. 2. Three vectors $\vec{A}$, $\vec{Q}$ and $\vec{K}$

Thus we obtain the following two results: (i) the total angular momentum **J** is parallel to the third axis, and (ii) the mesotronic angular momentum **L** is perpendicular to the spin angular momentum $\frac{1}{2}\tau$. In this manner, two vector **L** and $\frac{1}{2}\tau$ which are perpendicular to each other, precess around their resultant **J** This situation is indicated by Fig. 1

It is noteworthy that this relation between three vectors $\frac{1}{2}\tau$, **L** and **J** corresponds just to the relation between three vectors, electronic angular momentum, rotational angular momentum and total angular momentum of a diatomic molecule. If $\vec{\Lambda}$, $\vec{O}$ and $\vec{K}$ denote respectively these angular momenta, then $\vec{\Lambda}$ and $\vec{O}$ are also perpendicular to each other and precess around their resultant $\vec{K}$, which is fixed in the space. See Fig. 2.*

This correspondence allows us to picture the behavior of the mesotron field and the nuclear particle by means of a gyroscopic model as we have done in the case of diatomic molecules.

In the case of a gyroscopic diatomic molecule the angular velocity ϕ of the precession is given by

$$\phi=\frac{|\vec{K}|}{I}, \tag{1·29}$$

and the rotational energy by

$$E_{\text{rot}}=\frac{1}{2I}(\vec{K}^2-\vec{\Lambda}^2) \tag{1·30}$$

if I denotes the moment of inertia of the molecule. We see that when we replace $\vec{K}$ by **J** (noting $|\mathbf{J}|=|m|$) and $\vec{K}$ by $\frac{1}{2}\tau$, then we obtain just the second term of (1·26). From this correspondence we can see that $l^2\boldsymbol{K}_3$ plays the role of the moment of inertia I, and the second term in (1·26) is to be interpreted as the rotational energy of the system. This inertia $l^2\boldsymbol{K}_3$ is caused by the inertial reaction of the field.

The analogy with the diatomic molecule makes it possible to improve our formula (1·26), which has been obtained in a purely classical way, so as to fit better the quantum theory. For this purpose we have to replace m^2 by $J(J+1)$. Thus we had better to use

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* See, for instance, R. Mulliken: Band spectra, part II § 6, Rev. Mod. Phys. 2 (1930).

$$\left.\begin{aligned} \overline{\cos^2\theta}&=\frac{1}{4J(J+1)}, \quad \omega=\frac{\sqrt{J(J+1)}}{l\boldsymbol{K}_3} \\ H&=-\frac{l^2}{2}\boldsymbol{K}_1^3+\frac{1}{2l^2\boldsymbol{K}_3}\left\{J(J+1)-\frac{1}{4}\right\} \end{aligned}\right\} \qquad (1\cdot31)$$

in place of (1·26).

As stated above, the angular velocity ω is very small when the coupling is strong. This is due to the fact that in this case the moment of inertia $l^2\boldsymbol{K}_3$ is very large. Also in the case of diatomic molecules the angular velocity of the precession is very small. This situation has enabled one to treat the electronic motion in a diatomic molecule separately from the rotational motion of the molecule as a whole. The electronic angular momentum $\vec{\Lambda}$ is thus regarded as a "good" quantum number. The analogous situation makes it possible to treat also our problem of interaction between mesotron and nuclear particle quantum-theoretically. Such treatment is the method of Wentzel which will be described later. There we shall see that the first term of (1·29) just corresponds to the electronic energy of a diatomic molecule

Thus far we have adopted the symmetrical theory. When we will deal with the charged mesotron theory, we have only to put $c=0$ in (1·22) Then we find. instead of (1·22),

$$\left.\begin{aligned} a(\mathfrak{k})&=\frac{1}{K-\omega}\,\frac{l}{2}g(k)\sin\theta\, e^{-i\omega t} \\ b(\mathfrak{k})&=-\frac{1}{K+\omega}\,\frac{l}{2}g(k)\sin\theta\, e^{i\omega t} \end{aligned}\right\} \qquad (1\cdot32)$$

and, when the coupling is strong, we obtain in the charged mesotron theory

$$\left.\begin{aligned} \cos\theta&=\frac{m}{2l^4\boldsymbol{K}_1^3\boldsymbol{K}_3} \qquad \omega=\frac{m}{l^2\boldsymbol{K}_3} \\ H&=-\frac{l^2}{2}\boldsymbol{K}_1^3+\frac{1}{2l^2\boldsymbol{K}_3}m_2 \end{aligned}\right\} \qquad (1\cdot33)$$

in place of (1·26).

The main difference between the behavior of field and particle in these two theories is as follows: while in the symmetrical theory we have $\cos\theta=\frac{1}{2\cdot\frac{1}{2}}=1$ in the proton state [see (1·26)], we have in the charged theory $\cos\theta\cong0$ [see (1·33)]. Thus in the proton state the vector τ makes a slow precession lying almost in the equatorial plane in the latter theory, whereas it is parallel to the third axis and does not move in the former theory.

In the expression for H of (1·26), (1·31) and (1·33) the term containing m^2 represents the dependence of the energy on the charge of the system. This term gives rise to a physical consequence which has not been obtained by the ordinary perturbation theory. This is namely the following fact: When the coupling is strong enough, there is the possibility of stable states with $|m| \geqq 3/2$ besides the proton state (i. e. the state with $m=1/2$) or the neutron state (i. e. the state with $m=-1/2$). When, for instance, the excitation energy of the states with $|m|=3/2$ from the ground states, which have $|m|=1/2$, is smaller than the rest energy $\varkappa$ of a mesotron, i. e. when we have

$$H_{|m|=3/2}-H_{|m|=1/2}<\varkappa\,, \tag{1·34}$$

then the states with $|m|=3/2$ are stable. This means physically that the nuclear particle with the observable charge 2 or -1 can exist without dissociating into a mesotron and an ordinary proton or neutron. Such nuclear particles can still decay emitting β–ray, but their life will be of the order of 10^{-8} sec, so that there is the possibility that they are observed if such particles do really exist.

This dependence of H on m can be obtained also quantum-theoretically by means of Wentzel's method. In fact, the possibility of such stable proton-isobars in case of strong coupling has been concluded first by Wentzel himself.

The following fact is to be added here: If one calculates the self-energy H of a nuclear particle by means of the perturbation theory, we obtain

$$H=\left\{\begin{array}{ll} -\frac{3}{2}l^2\boldsymbol{K}_1^3 & \text{for symmetrical theory} \\ -l^2\boldsymbol{K}_1^3 & \text{for charged theory,} \end{array}\right\} \tag{1·35}$$

while our results show that, when the coupling is strong, the main term of H is $-(l^2/2)\boldsymbol{K}_1^3$ for both symmetrical and charged theories. This means that in case of strong coupling the perturbation theory will give the erroneous results for the self energy, which are too large by the factor 3 and 2 respectively for the symmetrical and the charged mesotron theories, if one applies this procedure without respect the fact that this will give reliable results only when the coupling sufficiently weak.

The same situation occurs also when we will calculate the nuclear forces between nuclear particles: We obtain too large nuclear forces, if we calculate them by means of the perturbation theory when the coupling is not weak enough. More precise discussions concerning this point will be given later.

C:—Scattering of a mesotron.

It has become clear by the above consideration that, when the coupling is strong, a nuclear particle interacts with the surrounding mesotron field in a way considerablly different from the manner of interaction in the case of weak coupling. It is expected, therefore, that the mesotron and the nuclear particle will show a quite different behavior on colliding with each other from that expected by the perturbation theory.

The essential point in the consideration above is that we should picture our system as a gyroscope with the large moment of inertia $l^2\boldsymbol{K}_3$ when the coupling is strong. If so, it is expected that this large moment of inertia will affect appreciablly on the scattering of a mesotron wave by the nuclear particle and diminish the scattering cross-section: When an incident mesotron wave impinges on the nuclear particle, it will cause a forced vibration of the vector τ, and this vibration of τ in turn will disturb the surrounding field and give rise to the scattered wave. But the large moment of inertia of the system will thereby act as a large inertial resistance against the motion of τ so that the amplitude of its motion will be very small. Thus the amplitude of the scattered wave will be much smaller than the amplitude expected by the perturbation theory which neglects wholly the field reaction. Such a situation has been first pointed out by Heisenberg. We shall here deal with this problem more precisely.

In order to avoid unnecessary complications, we shall here take the charged mesotron theory. Then, as we have stated above, the isotopic spin τ makes a very slow precession lying almost in the equatorial plane. Therefore, we can consider without introducing a large error that this vector remains fixed at some position in the equatorial plane. Thus we may put in this approximation $\omega=0$ and $\theta=\pi/2$. We denote further the azymuth of this vector by φ.

Now, let $K_0/2\pi$ denote the frequency of the incident wave, $\mathfrak{k}_0$ its propagation vector and a its amplitude. The vector τ will oscillate around its unperturbed position when the incident wave acts on it. Corresponding to this situation we put

$$\left.\begin{aligned}\tau_1&=\cos\varphi+\varepsilon\sin\varphi\sin(K_0t+\eta)\\ \tau_2&=\sin\varphi-\varepsilon\cos\varphi\sin(K_0t+\eta)\\ \tau_3&=\qquad\varepsilon_3\end{aligned}\right\}\qquad(1\cdot36)$$

where ε and η represent respectively the amplitude and the phase of the forced vibration of τ. It will be seen afterwards that, owing to the large inertial resistance, the amplitude ε is very small compared with unity, so far as the frequency K_0 of the incident wave is small compared with the cut-off momentum $\boldsymbol{K}$. Thus we neglect higher powers of ε throughout our calculation.

On substituting $(1\cdot36)$ into the equations of motion for $a(\mathfrak{k})$ and $b(\mathfrak{k})$, which are obtained by putting $c(\mathfrak{k})=0$ in $(1\cdot9)$, we obtain equations for the amplitudes $a(\mathfrak{k})$ and $b(\mathfrak{k})$. We solve these equations imposing the boundary condition: there exist at infinity, besides the incident plane wave with the propagation vector $\mathfrak{k}_0$ and of the amplitude a which represents, for instance, the flow of positive mesotrons, only out-going spherical waves corresponding to the scattered positive and negative mesotrons.

Carrying out this calculation, noting that we may put $\varphi=0$ without losing generality, we find

$$\left.\begin{aligned}a(\mathfrak{k})&=(2\pi)^{3/2}a\,\delta(\mathfrak{k}-\mathfrak{k}_0)\,e^{-iK_0t}+\frac{l}{2}\frac{g(k)}{K}+\frac{l\varepsilon}{4}\frac{g(k)}{K+K_0}e^{i(K_0t+\eta)}\\ &\quad-\frac{l\varepsilon}{4}g(k)\left\{\frac{1}{K-K_0}+i\pi\delta(K-K_0)\right\}e^{-i(K_0t+\eta)}\\ b(\mathfrak{k})&=-\frac{l}{2}\frac{g(k)}{K}+\frac{l\varepsilon}{4}\frac{g(k)}{K+K_0}e^{i(K_0t+\eta)}\\ &\quad-\frac{l\varepsilon}{4}g(k)\left\{\frac{1}{K-K_0}+i\pi\delta(K-K_0)\right\}e^{-i(K_0t+\eta)}.\end{aligned}\right\}$$

$(1\cdot37)$

The first term on the right hand side of the expression for $a(\mathfrak{k})$ represents the incident plane wave, which, on transforming from our $\mathfrak{k}$-representation to the ordinary representation in the coordinate space, gives rise to the plane wave:

$$\text{incident plane wave}=a\,e^{i(\mathfrak{k}_0\mathfrak{r}-K_0t)}.\qquad(1\cdot38)$$

The last terms containing the factor $\{(1/K-K_0)+i\pi\delta(K-K_0)\}$ in the expressions for both $a(\mathfrak{k})$ and $b(\mathfrak{k})$ represent the outgoing spherical waves,* which transformed from our $\mathfrak{k}$–representation to the r–representation give rise to the spherical wave

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* See P.A.M. Dirac: Quantum Theory § 53.

$$\text{scattered spherical wave} = -\sqrt{2\pi}\,\frac{l\varepsilon}{2}\,K_0\,g(k_0)\,\frac{e^{i(k_0 r - K_0 t - \eta)}}{r}. \tag{1·39}$$

That the expression for $b(\mathfrak{k})$ does not contain the term $a\delta(\mathfrak{k}-\mathfrak{k}_0)$ corresponds to our assumption that the incident mesotron is of positive charge. One notices further that the scattered waves in both expressions for $a(\mathfrak{k})$ and $b(\mathfrak{k})$ have the same form. This fact means that the probability of the incident positive mesotron being scattered with the change of its sign is equal to the probability of the scattering without the change of the sign.

In order to determine the unknown phase η and the unknown amplitude ε, we now insert (1·37) into the equation of motion for τ, which is obtained in by putting $c(\mathfrak{k})=0$ in (1·10), and equate the coefficients of $\cos K_0 t$ and $\sin K_0 t$ on both sides. We obtain then

$$\left.\begin{aligned} \varepsilon &= \frac{l^2}{K_0}\left\{(2\pi)^{3/2} a\,\frac{l g(k_0)}{K_0}\cos\eta - \frac{2l^2\varepsilon}{K_0}\mathfrak{K}\right\}\boldsymbol{K}_1^3 \\ 0 &= -(2\pi)^{3/2} a\,\frac{l g(k_0)}{K_0}\sin\eta - 4\pi^2 k_0 l^2 g(k_0)^2\,\varepsilon \end{aligned}\right\} \tag{1·40}$$

where $\mathfrak{K}$ is the abreviation*

$$\mathfrak{K} = \int g(k)^2\left\{\frac{K}{K^2-K_0^2} - \frac{1}{K}\right\}d\mathfrak{k}. \tag{1·41}$$

The first equation of (1·40) gives now

$$(2\pi)^{3/3} a l g(k_0)\cos\eta = \frac{K_0}{l^2\boldsymbol{K}_1^3}\varepsilon + 2l^2\mathfrak{K}\,\varepsilon, \tag{1·42}$$

or, neglecting the first term on the right-hand side which is small compared with the second, we get

$$(2\pi)^{3/2} a l g(k_0)\cos\eta = 2l^2\mathfrak{K}\,\varepsilon. \tag{1·42'}$$

On eliminating η from this and the second equation of (1·40) we obtain

$$\varepsilon^2 = \frac{(2\pi)^3 a^2 l^2 g(k_0)^2}{l^4\mathfrak{K}^2 + \frac{1}{4}(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2}. \tag{1·43}$$

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* This German letter does not mean a vector.

Now, the scattering cross-section is given by

$$\text{cross-section} = 4\pi \frac{|r \text{ times the amplitude of the scattered spherical wave}|^2}{|\text{the amplitude of the incident plane wave}|^2} \tag{1·44}$$

By using (1·38) and (1·39) here, we get the cross-section σ_+ with which the incident positive mesotron is scattered without changing its sign as well as the cross-section σ_- with which the incident mesotron is scattered changing its sign:

$$\sigma_+ = \sigma_- = \frac{1}{4} \frac{\pi}{k_0^2} \frac{(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2}{l^4 \mathfrak{K}^2 + \frac{1}{4}(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2} . \tag{1·45}$$

If the frequency of the incident mesotron is so small that we have $K_0 \ll \boldsymbol{K}$, we can put approximately

$$l^2 \mathfrak{K} = (l^2 \boldsymbol{K}_3) K_0^2 . \tag{1·46}$$

In this case (1·45) is simplified into

$$\sigma_+ = \sigma_- = \frac{1}{4} \frac{\pi}{k_0^2} \frac{(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2}{(l^2 \boldsymbol{K}_3)^2 K_0^4 + \frac{1}{4}(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2} . \tag{1·47}$$

We give here, for the comparison, the corresponding cross-section obtained by the perturbation theory applied to our problem:

$$\left.\begin{aligned} \sigma_+ &= \frac{4\pi}{k_0^2} \frac{(4\pi^2 l^2 g(k_0)^2 K_0 k_0)^2}{K_0^2} \\ \sigma_- &= 0 . \end{aligned}\right\} \tag{1·48}$$

The difference between (1·47) and (1·48) is as follows: According to (1·48) the incident mesotron does never change its sign on colliding with a nuclear particle, whereas (1·47) shows that the incident mesotron is scattered changing its sign as well as without changing its sign with the same probability. Further, the cross-section (1·47) differs substantially from the cross-section (1·48) in that the former does not increase indefinitely with increasing energies of the incident particle but approsches to the asymptotic value π/k_0^2, thus without exceeding the square of the wave length of the incident particle.

This difference of (1·47) from (1·48) represents the effect of the field reaction. The two terms in the denominator of (1·47) correspond each to the two kinds of the field reactions, the inertial on one hand the damping on the other hand. Indeed, the first term contains the factor $l^2\boldsymbol{K}_3$ which is the moment of inertia of the system due to the self-field of the particle. That the second term represents the effect of damping can be seen by tracing our course of calculation. One sees then that this term has appeared in connection with the term $i\pi\delta(K-K_0)$ in (1·37). This means that this second term is connected directly with our boundary condition saying that there exist only outgoing waves at infinity. That the second term corresponds to the damping reaction of the field is implied also by the fact that (1·47) has the form of a dispersion formula and thereby this term plays the role of the width of the resonance level.

It is here to be noted that the second term in the denominator of (1·47) is small compared with the first term, so far as $K_0 \ll \boldsymbol{K}$ holds. This means that in such a case the effect of the damping is small compared with the effect of the inertia. This shows in turn that the ampletude of the forced vibration of τ is small compared with unity, because if it were not the case, the damping reaction would be necessarily large so that the second term would predominate the first. Our assumption $\varepsilon \ll 1$ can be thus justified.

In conclusing the classical treatment, we shall notice that every result obtained in this paragraph agrees perfectly with the corresponding result obtained quantum-theoretically by means of Wentzel's method. This will be shown in the next paragraph. The physical consequences of our results will be described in the last paragraph.

(to be continued).

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A Note on the Method of Radiation Damping and its Applicatio to the Photo-mesonic Process.*

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1. As has been shown by Heilter, Wilson and others,(1) a part of the converging difficulties, which is connected with the special form of the interaction energies, in the calculation of the transition probabilities can be removed by the method of 'radiation damping,' which was first introduced by Waller(2) into quantum mechanical calculations. In the scattering processes, the cross sections increase unlimitedly when the interaction energies between the scattering particles and incoming (or outgoing) fields increase with the increasing energies of incoming quanta. But, in such cases the transition probabilities of the inverse processes also tend to infinity according as the probabilities of the scattering processes diverge. And, we can expect that these inverse processes suppress the increasing of the scattering probabilities as in the case of the classical theory of radiation damping. The calculations made by the authors cited above have shown that this is the case for the scattering of mesons by nucleons. But, we think the physical meaning of this method is somewhat obscurely shown in such an example. In this paper, we shall inquire into the reason why this method generally leads to the converging results and illustrate it with the application to the photo-mesonic process where the final state is defferent in kind from the initial state.

2. Let us consider the transition process between the initial state i and the final state f, and denote the matrix element connecting these two states by H_{if}. There may be many kinds of states f' which are connected with the initial state i by non vanishing matrix elements $H_{if'}$. One must solve the simultaneous equations for the probability amplitudes of these states in order to calculate the transition probabilities from the initial state to these

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* The contents of this paper were reported at the semi-annual meeting of I.P.C.R. held June 1943, but the publication has been postponed on account of the war.

(1) W. Heitler: Proc. Cambridge Phil. Soc., 38 (1941), 291.
A. H. Wilson; ibid., 38 (1941), 301.

(2) I. Waller: Zeits. für Physik, 88 (1934), 436.

final states. Here, we must take into account all these final states even if whose energies are different from that of the initial one. As is well known, the summation (or integration) of the transition probabilities over these states which are different in energy leads to the approximate conservation of energy. On the other hand, if we demand that the inverse processes shall be treated on an equal footing with the transition processes in order to take the reaction effects into account, the transitions from final states to the states i' which are connected with final states f' (including f) by $H_{f'i'}$ and belong to the same class as the initial one but different from it in energy must be involved in the simultaneous equations, and we must perform the summation (or intergration) over these states. One will see in the following formulations that this summation just secures the convergence of the transition probability for any process caused by the interaction terms which increase unlimitedly with increasing energy.

When initial and final states are of the same kind as in the case of pure scattering (where the incoming and outgoing particles are of the same kind) which has been picked up by Heitler and Wilson as an example, the distinction between these two kinds of summations disappears, and this circumstance confuses the physical interpretation of the converging effect of the method of radiation damping.

3. Next, we shall formulate this method according to Wilson's paper,$^{(1)}$ but making distinction between two cases mentioned above, i. e. *case 1.* the initial state is different in kind from the final states and *case 2.* they are of the same kind.

In *case 1*, the simultaneous equations for the probability amplitudes are given as follows

$$\left.\begin{aligned} -i\hbar\,\dot{a}_i &= \sum_{f'} H_{if'}\,a_{f'}\,e^{i(E_{f'}-E_i)t/\hbar} \\ -i\hbar\,\dot{a}_{i'} &= \sum_{f'} H_{i'f'}\,a_{f'}\,e^{i(E_{f'}-E_{i'})t/\hbar} \\ -i\hbar\,\dot{a}_{f'} &= H_{f'i}\,a_i\,e^{i(E_i-E_{f'})t/\hbar}+\sum_{i'} H_{f'i'}\,a_{i'}\,e^{i(E_{i'}-E_{f'})t/\hbar} \end{aligned}\right\} \qquad (1)$$

$$(H_{if'}=\widetilde{H}_{f'i},\quad H_{i'f'}=\widetilde{H}_{f'i'})$$

where a_i, $a_{i'}$ and $a_{f'}$ are the probability amplitudes of the initial state i with energy E_i, those of states i with energies $E_{i'}$ which are of the same kind as the initial state, and those of the different final states f' (including

f) with energy $E_{f'}$ respectively, and $\dot{a}_i$,......denote the differential coefficients of a_i,......with respect to time t.

In *case 2*, the simultaneous equations (1) reduced to

$$\left.\begin{aligned} -i\hbar a_i &= \sum_{f'} H_{if'}\, a_{f'}\, e^{i(E_{f'}-E_i)t/\hbar} \\ -i\hbar \dot{a}_{f'} &= H_{f'i}\, a_i e^{i(E_i-E_{f'})t/\hbar} + \sum_{f''} H_{f'f''}\, a_{f''}\, e^{i(E_{f''}-E_{f'})t/\hbar} \end{aligned}\right\} \qquad (2)$$

where $\sum\limits_{f''}$ denotes the summation which is taken all over other final states (or, strictly speaking, initial and final states) f'' than f'. These equations are identical with those obtained by Wilson and others. Here, it must be noted that the appearance of the matrix elements $H_{f'f''}$ which connect two final states is due to the accidental circumstance that the initial and final states are of the same kind in this case.

4. When the process takes place through the intermediate states n, n'......, we have to comprehend that the matrix elements $H_{if'}$ (or $H_{f'f''}$) represent those composed by performing the summation $\sum\limits_n \dfrac{H_{in}H_{nf'}}{E_i-E_n}$, $\sum\limits_n\sum\limits_{n'} \dfrac{H_{in}H_{n'n}H_{n'f'}}{(E_i-E_n)(E_i-E_{n'})}$, ...all over the intermediate states n, n',... . Then, they are given in the form

$$H_{if'} = H_{if'}\,(\text{direct}) + \sum_n \frac{H_{in}H_{nf'}}{E_i-E_n} + \sum_n\sum_{n'} \frac{H_{in}H_{nn'}H_{n'f'}}{(E_i-E_n)(E_i-E_n')} + \cdots\cdots \qquad (3)$$

For these summations with respect to the intermediate states, we must content ourselves to obtain the first non vanishing terms and omit the diverging summations which correspond to the self-energies of particles concerning in transition processes. Then, as will be shown in the following discussions, in no case does the transition probability diverge as long as the inverse processes are taken into account correctly. This is very satisfactory result, because as is generally believed and discussed by us in other places(3) those diverging terms rising from the fact that the self-forces of elementary particles become infinity when the point models are taken for these particles must be removed entirely from the expression for the energy of any system.

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(3) M. Kobayasi and E. Kanai: Progress of Theoretical Physics, **1** (1946), 21, and other papers shortly published by us in this journal.

5. In order to solve the simultaneous equations (1), we put

$$\left.\begin{aligned} a_i &= e^{-\frac{1}{2}\Gamma t} \\ a_{f'} &= A_{f'} \frac{e^{i(E_i - E_{f'} + \frac{1}{2} i\hbar\Gamma)t/\hbar} - 1}{E_i - E_{f'} + \frac{1}{2} i\hbar\Gamma} \\ a_{i'} &= A_{i'} \frac{e^{i(E_i - E_{i'} + \frac{1}{2} i\hbar\Gamma)t/\hbar} - 1}{E_i - E_{i'} + \frac{1}{2} i\hbar\Gamma} \end{aligned}\right\} \qquad (4)$$

in accordance with initial conditions

$$a_i = 1, \ a_{f'} = a_{i'} = 0 \quad \text{for} \quad t = 0. \qquad (5)$$

and substitute these expressions in (1). Then we obtain the equations to determine the unknown quantities Γ, $A_{f'}$ and $A_{i'}$ in (4). By replacing the summations with respect to states f' and i' by integrals in these equations, and using the well-known formula

$$\int f(E) \frac{1 - e^{i(E_0 - E + i\gamma)t/\hbar}}{E_0 - E + i\gamma} dE = i\pi f(E_0)^{(4)}$$

we get

$$\left.\begin{aligned} \frac{1}{2}\Gamma &= \frac{\pi}{\hbar}\left[\int H_{if'} A_{f'} \rho_{f'} d\Omega_{f'}\right]_{E_{f'} = E_i} \\ A_{f'} &= H_{f'i} + (i\pi)^2 \left[\iint H_{f'i'} H_{i'f''} A_{f''} \rho_{i'} \rho_{f''} d\Omega_{i'} d\Omega_{f''}\right]_{E_{f''} = E_{i'} = E_i} \\ A_{i'} &= i\pi \left[\int H_{i'f'} A_{f'} \rho_{f'} d\Omega_{f'}\right]_{E_{f'} = E_i} \end{aligned}\right\} \qquad (6)$$

where $\rho_{f'}$ and $\rho_{i'}$ represent the densities of final and initial states with respect to energy, and $\int d\Omega_{f'}$ and $\int d\Omega_{i'}$ denote the integrals over other freedoms of states than energy, for example the solid angles into which the particles are scattered. For *case 2*, $A_{f'}$ is given simply by

$$A_{f'} = H_{f'i} + i\pi\left[\int H_{f'f''} A_{f''} \rho_{f''} d\Omega_{f''}\right]_{E_{f''} = E_i} \qquad (7)$$

in accordance with Wilson's result.

By solving these integral equations for $A_{f'}$, we get for the transition probabilities or the differential cross-sections for the scattering processes

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(4) In derivation of this formula one must assume that $f(E)$ converges sufficiently for large E. In this point the present formulation is not yet free from difficulties.

$$d\phi_f = \frac{2\pi}{\hbar c}\rho_f |A_f|^2 d\Omega_f. \tag{8}$$

For the pure scattering where the incident and scattered particles are of the same kind, we can use the integral equation (7) for A_f. But, when the incident particle is different from the outgoing particle, as in the case of photo-mesonic process where the incident photon is absorbed by nuclear particles with subsequent emission of charged meson, we must substitute for A_f, the expression which is given by solving the equation (6)

As is easily seen from the form of the integral equations (6) and (7), the results thus obtained always converge in both cases even when matrix elements H_{if} deverge with increasing energy.

6, Now, we shall apply above results for the *case I* to the photo-mesonic effect which was already treated by Heitler⁽⁵⁾ and by us⁽⁶⁾ by making use of the ordinary perturbation method, and the results obtained in this way strongly diverge contrary to the cosmic ray evidences. In the previous paper, we have avoided this difficulty by taking recourse to cutting-off prescription. But, when the method of radiation damping is applied the cross-section for this process decreases rapidly and one of the discrepancies between the meson theory and experiments disappears as a reasonable consequence.

First, we adopt the pseudo-scalar wave functions for mesons. Then the matrix element for this transition process is approximately given by

$$H_{if} = -\frac{2\pi ef}{\mu c^2}\frac{\hbar^2 c^2}{\sqrt{E_0 E_k}}\tilde{u}(0)(\sigma\cdot\boldsymbol{e}_0)u(\boldsymbol{k}) \tag{9}$$

as long as the mass of nucleon is assumed to be infinite, where $u(\boldsymbol{k})$ denote the wave function of the nucleon with momentum $\boldsymbol{k}$, $\boldsymbol{e}_0$ is the unite vector in the direction of polarization of incident photon and σ is the spin operator of nucleon. Further, f is the interaction constant between nucleon and meson field, μ the meson mass, and E_0, E_k are the energies of incident photon and emitted meson respectively. The solutions of the integral equation (6) are given by the form

$$A_f = a\tilde{u}(\boldsymbol{k})(\boldsymbol{e}_0\cdot\sigma)u(0), \tag{10}$$

and by making use of equations (6) and (8) we obtain

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(5) W. Heitler: Proc. Roy. Soc. A, **166** (1938), 529.

(6) M. Kobayasi and T. Okayama: Proc. Phys-Math. Soc. Japan, **21**, (1939), 1.

$$a=\frac{A_f^0}{1+32\pi\rho_i\rho_f|A_f^0|^2},\quad A_f^0=-2\pi\frac{ef}{\mu c^2}\frac{(\hbar c)^2}{\sqrt{E_0E_k}},\tag{11}$$

and

$$|A_f|^2=\frac{|A_f^0|^2}{(1+32\pi^4\rho_i\rho_f|A_f^0|^2)^2},\tag{12}$$

where $\rho_f\doteqdot\rho_i=\frac{l_0^2}{(2\pi)^3\hbar c}$, and $\hbar l_0\left(l_0\gg\frac{\hbar}{\mu c}\right)$ is the momentum of incident photon. The differential cross-section for the emission of meson in the solid angle $d\Omega$ is finally given by

$$d\phi_f=\frac{R^2}{(1+2R^2l_0^2)^2}d\Omega\tag{13}$$

with

$$R^2=\left(\frac{ef}{\mu c^2}\right)^2$$

And, the total cross-section for photo-mesonic effect is given by

$$\phi_f=\frac{4\pi R^2}{(1+R^2l_0^2)^2}\tag{14}$$

It is remarkable that this cross-section decreases with increasing energy E_0 of incident photon as $1/E_0^4$ in the high energy region, whereas in the example belongs to *case 2* it decreases as $1/E_0^2$.

By using our formulations, we can also calculate the probability for the transition process from the initial state i to other states i' which belong to the same class as the initial one. In the present example, this transition corresponds to the pure scattering of photon by nucleon caused by the interaction between photons and virtually emitted mesons. The cross-section for this process is calculated by making use of $A_{ii'}$ and given by

$$d\phi_{ii'}=\frac{2\pi}{\hbar c}\rho_{i'}|A_{ii'}|^2d\Omega_i=R^2l^2d\phi_f\tag{15}$$

and

$$\phi_{ii'}=l_0^2R^2\phi_f.\tag{16}$$

For the vector meson theory, we can calculate the cross-section for the photo-mesonic process in the similar way to the above case by making use of the matrix elements obtained by us in the previous paper(5). Here, we omitt the detailed expressions for the matrix elements in this case and write down the results obtained. Though the integral equations can not be solved exactly in this case, we can estimate the value A_f by replacing $|H_{if}|^2$ by the average value $|\overline{H_{if}}|^2$. Then A_f is given approximately by

$$A_f = \frac{H_{fi}}{1+16\pi^4 \rho_i |\overline{H_{if}}|^2}.$$

For the cross-section we obtain

$$\phi_f = \frac{\phi_0}{[1+(a/4\pi)\phi_0 l_0^2]^2}, \tag{17}$$

where ϕ_0 denotes the cross-section obtained by neglecting the radiation damping, and $a=1$ or 2 according as the direction of polarization of the emitted meson parallel or perpendicular to its momentum. As has been shown in the previous paper, ϕ_0 itself increases unlimitedly with the energy of incident photon, but the expression (17) always converges with increasing energy E_0 as $1/E_0^4$. This is a general law which holds for the scattering processes where the incident and scattered particle are different in kind.

From the above calculations, we can conclude that when the reactions of fields on the particles are taken into account correctly, the diverging difficulties concerning with the transition probability for any process disappear as long as the infinite self-energies of elementary particles are laid asleep.

In conclusion we wish to express our cordial thanks to Professor H. Yukawa for his encouragement and interest in this paper. Further, we are indebted to Mr. E. Kanai for his kind discussions on this subject.

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On the Effect of the Field Reactions on the Interaction of Mesotrons and Nuclear Particles, II.

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§ 2. Wentzel's method.(4),(5)

As stated in the introduction, it is also possible to treat the problem quantum-theoretically when the coupling is Strong. This consist in obtaining the solution in an expansion of inverse powers of the coupling constant l. Such a method was first introduced by Wentzel, which we shall here first modify so as to be applied to our problem without the inconvenient and artificial use of the lattice space. We shall also make the underlying idea of this method more clear by means of the analogy of our system with a diatomic molecule.

A :—Separation of three motions.

In order to simplify the problem, we adopt also here the charged mesotron theory. Further, we use the real amplitudes $\xi_1(\mathfrak{k})$ and $\xi_2(\mathfrak{k})$ and their canonically conjugate momenta $p_1(\mathfrak{k})$ and $p_2(\mathfrak{k})$ ($\xi_3(\mathfrak{k})$ and $p_3(\mathfrak{k})$ do not appear, as we are dealing with the charged theory). The Hamiltonian function is then written as

$$H=\frac{1}{2}\int\left\{\mathbf{p}(\mathfrak{k}^2)+K^2\boldsymbol{\xi}(\mathfrak{k})^2\right\}d\mathfrak{k}-l\int g(k)\sqrt{K}\,(\boldsymbol{\xi}(\mathfrak{k})\cdot\tau)d\mathfrak{k}\,, \tag{2. 1}$$

when we use a thick letter to represent a vector in the 12-space.

We introduce now a complete set of normalized orthogonal functions having $g(k)\sqrt{K}$ as the zero-th member:

$$\varphi_0(\mathfrak{k})=\frac{1}{\sqrt{\boldsymbol{K}_{-1}^{5}}}g(k)\sqrt{K}\,,\ \varphi_1(\mathfrak{k}),\ \varphi_2(\mathfrak{k}),\ \ldots\ldots,\ \varphi_s(\mathfrak{k}), \tag{2. 2}$$

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(4) G. Wentzel: Helv. Phys. Acta. 13 (1940), 269.

(5) J. R. Oppenheimer and T. Schwinger: loc. cit.

We assume here that all members are real.

We expand then $\boldsymbol{\xi}(\mathfrak{x})$ and $\mathbf{p}(\mathfrak{x})$ in terms of these orthogonal functions:

$$\boldsymbol{\xi}(\mathfrak{x})=\sum_{s=0}^{\infty}\boldsymbol{\xi}_s\varphi_s(\mathfrak{x}),\quad \mathbf{p}(\mathfrak{x})=\sum_{s=0}^{\infty}\mathbf{p}_s\varphi_s(\mathfrak{x}) \tag{2. 3}$$

and regard the coefficients ξ_0, ξ_1, ξ_2, …… as a new set of coordinates, and $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, …… as the corresponding set of momenta. In terms of these new canonical variables the Hamiltonian becomes

$$H=\frac{1}{2}\sum_{s=0}^{\infty}\mathbf{p}_s^2+\frac{1}{2}\sum_{r,s}c_{r,s}(\boldsymbol{\xi}_r\cdot\boldsymbol{\xi}_s)-\sqrt{\boldsymbol{K}_{-1}^{5}}\,l(\boldsymbol{\xi}_0\cdot\boldsymbol{\tau}), \tag{2. 4}$$

in which the coefficients $c_{r,s}$ are defined by

$$c_{r,s}=\int\varphi_r(\mathfrak{x})K^2\varphi_s(\mathfrak{x})d\mathfrak{x}. \tag{2. 5}$$

Now, $\boldsymbol{\xi}_0$ represents so to speak the intensity of the field at the origin. Its order of magnitude can be roughly determined by the classical consideration and it is found that

$$\langle|\boldsymbol{\xi}_0|\rangle_{av}\cong l\boldsymbol{K}^{1/4} \tag{2. 6}$$

By the classical consideration in the preceding paragraph we know further that $\boldsymbol{\xi}_0$ rotates in the 12-space with a small angular velocity. This slowness of the rotation results in the quantum theory in that the vector $\frac{1}{2}\boldsymbol{\tau}$ will be quantized in the first approximation parallel or antiparallel to the direction $\boldsymbol{\xi}_0$. This situation resembles to the case of a diatomic molecule, in which the angular momentum of the electronic motion is quantized with respect to the direction joining the nuclei and its component along this direction is considered as a "good" quantum number, The electronic states of the molecule can be thus specified by this quantum number. In our system there are two states, the state of parallel spin and the state of antiparallel spin, and each corresponds to the electronic state of the molecule.

In the theory of diatomic molecules one knows that three motions, the motion of the inner electrons, the vibrational motion of the constituent nuclei and the rotational motion of the molecule as a whole, can be treated separately. This possibility has its ground in that the frequencies of these three motions differ so much from one another, that in dealing with the motion of the electrons which has a very high frequency the vibrational and the rotational motion can be regarded as an adiabatic change, in dealing with the vibrational motion which has an intermediate frequency the rotation of the molecule can also be regarded as taking place adiabatically.

Similar situation occurs also in our system of mesotrons and nuclear particle, only if the coupling is strong. Our Hamiltonian (2, 1) shows first that the energy difference between two spin states (i.e. the state in which $\frac{1}{2}\tau$ and ξ_0 are parallel and the state in which they are antiparallel) will roughly be given by

$$\varDelta E_{spin} \cong \sqrt{\boldsymbol{K}_{-1}^{5}}\, l \,\langle\, |\xi_0| \,\rangle_{av} \cong l^2\boldsymbol{K}^2 \tag{2. 7}$$

and this means that the frequency of the “ inner motion of the τ-spin has an order of magnitude of $l^2\boldsymbol{K}^2$. Consider next the various oscillating modes of the mesotron field interacting with the nuclear particle. Then they have frequencies from $\varkappa$ to $\boldsymbol{K}$ according as their wave length, thus

$$\varDelta E_{vib} \cong \varkappa \sim \boldsymbol{K} \tag{2. 8}$$

Now, we have seen in the preceding paragraph that there is one more mode of motion, i.e. the precession of the system. This motion has the frequency of the order of $1/l^2\boldsymbol{K}^2$ as indicated by the m-dependent term in (1, 33):

$$\varDelta E_{rot} \cong \frac{1}{l^2\boldsymbol{K}^2} \tag{2. 9}$$

If we now assume that the coupling is so strong that we have

$$l^2\boldsymbol{K}^2 \gg 1, \quad l^2\boldsymbol{K}^2 \gg \boldsymbol{K}/\varkappa, \tag{2. 10}$$

then the frequencies of these three kinds of motion lie quite apart from one another. In such a case three motions, the inner spin motion, the vibrational motion of the field, and the rotational motion of the system can be treated separately in the way similar to the familiar theory of diatomic molecules.

In order to describe the rotational motion of a molecules as a whole one has imagined a coordinate axis fixed to the molecule and has introduced the angle of rotation of this moving axis to specify the orientation of the molecule in the space. The electronic motion in the molecule has been then described referring to this moving axis. The analogy of our system to the diatomic molecule suggests now also to introduce the coordinate axis rotating with ξ_0. We introduce thus the ξ'-axis in the direction of ξ_0 (this direction corresponds to the intramolecular axis for the diatomic molecule), and the ξ''-axis in the direction perpendicular to it. Let θ be the angle of rotation of the ξ'-axis from the fixed ξ_1-axis. This angle is regarded as the coordinate describing the rotation of the system.

We write now the s-th coordinates referred to the new axis by ξ_s' and ξ_s'', then they are related to the old coordinates ξ_{s1} and ξ_{s2} by means of

$$\begin{cases} \xi_{01}=\xi_0' \cos\theta \\ \xi_{02}=\xi_0' \sin\theta \\ \text{(We write sometimes } \rho \text{ instead of } \xi_0') \end{cases} \tag{2. 11}$$

and

$$\begin{cases} \xi_{s1}=\xi_s' \cos\theta-\xi_s'' \sin\theta \\ \xi_{s2}=\xi_s' \sin\theta+\xi_s'' \cos\theta. \end{cases} \tag{2. 12}$$

If we denote the momenta conjugate to ξ_s' and ξ_s'' by p_s' ane p_s'', and let the momentum conjugate to θ be p_θ, then we have

$$\begin{cases} p_{01}=p_0' \cos\theta-\dfrac{\sin\theta}{\rho}p_\theta+\dfrac{\sin\theta}{\rho}\sum_s{}' (\xi_s'p_s''-\xi_s''p_s') \\ p_{02}=p_0' \sin\theta-\dfrac{\cos\theta}{\rho}p_\theta-\dfrac{\cos\theta}{\rho}\sum_s{}' (\xi_s'p_s''-\xi_s''p_s') \\ \text{(We write sometimes } p_\rho \text{ instead of } p_0') \end{cases} \tag{2. 13}$$

and

$$\begin{cases} p_{s1}=p_s' \cos\theta-p_s'' \sin\theta \\ p_{s2}=p_s' \sin\theta+p_s'' \cos\theta, \end{cases} \tag{2. 14}$$

where the dashed summation $\sum_s{}'$ means that the term with $s=0$ is not included.

In terms of these new coordinates and their conjugate momenta the Hamiltonian is expressed as

$$H=\frac{1}{2}\left[p_\rho^2+\frac{1}{\rho}p_\rho+\frac{1}{\rho^2}\left\{p_\theta-\sum_s{}' (\xi_s'p_s''-\xi_s''p_s')\right\}^2\right]$$

$$+\left\{\sum_s{}' p_s'^2+\sum_{r,s} c_{r,s}\xi_r'\xi_s'\right\}+\frac{1}{2}\left\{\sum_s{}' p_s''^2+\sum_{r,s}{}' c_{r,s}\xi_r''\xi_s''\right\}$$

$$-\sqrt{\boldsymbol{K}_{-1}^5}\, l\rho\begin{pmatrix}0 & e^{-i\theta}\\ e^{i\theta} & 0\end{pmatrix}, \tag{2. 15}$$

we have used here the representation

$$\tau_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2. 16}$$

Our problem is now to solve the eigen-value problem

$$(H-K_0)\,\Psi=0. \tag{2. 17}$$

The analogy to the theory of molecules* shows that this equation can be solved approximately by putting

$$\Psi=\Psi_{spin}\cdot\frac{1}{\sqrt{\rho}}P(\rho,\xi)\,\Theta(\theta), \tag{2. 18}$$

thereby the wave function Ψ_{spin} describing the inner spin motion satisfies the equation

$$\left\{-\sqrt{\boldsymbol{K}_{-1}^5}\, l\rho\begin{pmatrix}0 & e^{-i\theta}\\ e^{i\theta} & 0\end{pmatrix}-W(\rho)\right\}\Psi_{spin}=0, \tag{2. 19}$$

in which ρ and θ is considered as the parameters describing the adiabatic motion; the vibrational rotational part $P\Theta$ satisfies then the equation

$$\left\{\frac{1}{2}\left[p_\rho^2-\frac{1}{4\rho^2}+\frac{1}{\rho^2}\left\{p_\theta-\sum_s{}'(\xi_s'p_s''-\xi_s''p_s')\right\}^2\right]+\frac{1}{2}(\Sigma' p_s'^2+\sum_{r,s} c_{r,s}\xi_r'\xi_s')\right.$$

$$\left.+\frac{1}{2}(\sum_s{}' p_s''^2+\sum_{r,s}{}' c_{r,s}\xi_r''\xi_s'')+W(\rho)+\bar{U}-K_0\right\}P(\rho,\xi)\Theta(\theta)=0 \tag{2. 20}$$

in which the eigen-value $W(\rho)$ of (2, 19) plays the role of potential energy and $\bar{U}$ is the operator defined by

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* See, for instance, R. L. Kronig · Band Spectra and Molecular Structure,

$$\bar{U}=-\frac{1}{\rho^{2}}\Big\{-\psi_{1}{}^{*}\frac{\partial^{2}\psi_{1}}{\partial\theta^{2}}-\psi_{2}{}^{*}\frac{\partial^{2}\psi_{2}}{\partial\theta^{2}}-\frac{2\psi_{1}{}^{*}}{\Theta}\frac{\partial\psi_{1}}{\partial\theta}\frac{\partial\Theta}{\partial\theta}-\frac{2\psi_{2}{}^{*}}{\Theta}\frac{\partial\psi_{2}}{\partial\theta}\frac{\partial\Theta}{\partial\theta}$$

$$-\frac{2}{i}\sum_{s}{}'(\xi_{s}{}'p_{s}{}''-\xi''p_{s}{}')\left(\psi_{1}{}^{*}\frac{\partial\psi_{1}}{\partial\theta}+\psi_{2}\frac{\partial\psi_{2}}{\partial\theta}\right)\Big\} \tag{2. 21}$$

In (2. 21) ψ_1 and ψ_2 are two rows of Ψ_{spin} which is normalized: we have namely

$$|\psi_1|^2+|\psi_2|^2=1 \tag{2. 22}$$

when Ψ_{spin} is written in the form

$$\Psi_{spin}=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{2. 23}$$

Now from (2, 19) we obtain two eigen-functions

$$\Psi_{spin}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e_{i\theta}\end{pmatrix} \quad\text{and}\quad \Psi_{spin}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -e_{i\theta}\end{pmatrix} \tag{2. 24}$$

corresponding respectively to the eight-value

$$W(\rho)=-\sqrt{\boldsymbol{K}^{5}_{-1}}\,l\rho \quad\text{and}\quad W(\rho)=\sqrt{-\boldsymbol{K}^{5}_{-1}}\,l\rho \tag{2. 25}$$

But as we are dealing with the lowest states, we have to take only the solution with the lower energy:

$$\Psi_{spin}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\theta}\end{pmatrix},\quad W(\rho)=-\sqrt{\boldsymbol{K}^{s}_{-1}}\,l\rho. \tag{2. 26}$$

(it is seen that in this lower state the vector $\frac{1}{2}\tau$ is quantized parallel to ξ_0).

From (2, 21) we obtain immediately the rotational part

$$\Theta(\theta)=e^{i(m-\frac{1}{2})\theta},\qquad m=\ldots\ldots-\frac{2}{3},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{2}{3},\ \ldots\ldots \tag{2. 27}$$

Inserting the spin and the rotational parts thus obtained into (2,20), we obtain the equation for the vibrational part:

$$\Big[\frac{1}{2}p_{\rho}^{2}+\frac{1}{2}\Big\{\sum_{s}{}'p_{s}{}'^{2}+\sum_{r,s}c_{r,s}\xi_{r}{}'\xi_{s}{}'\Big\}-\sqrt{\boldsymbol{K}^{5}_{-1}}\,l\rho+\frac{1}{2}\Big\{\sum_{s}{}'p_{s}{}''^{2}$$

$$+\sum_{r,s}{}'c_{r,s}\xi_{r}{}''\xi_{s}{}''\Big\}+\frac{1}{2\rho^{2}}\Big\{m-\sum_{s}{}'(\xi_{s}{}'p_{s}{}''-\xi_{s}{}''p_{s}{}')\Big\}^{2}-K_{0}\Big]P\,\rho,\xi)=0, \tag{2. 28}$$

which will be solved in the following way.

B :—Determination of eigen-values.

We proceed now to the eigen-value problem (2. 28). We put

$$\begin{cases} H' = \dfrac{1}{2}p_\rho^2 + \dfrac{1}{2}\left\{\sum_s{}' p_s'^2 + \sum_{r,s} c_{r,s}\xi_r'\xi_s'\right\} - \sqrt{\boldsymbol{K}_{-1}^5}\, l\xi_0' \\ \quad = \dfrac{1}{2}\sum_s p_s'^2 + \dfrac{1}{2}\sum_{r,s} c_{r,s}\xi_r'\xi_s' - \sqrt{\boldsymbol{K}_{-1}^5}\, l\xi_0' \\ H'' = \dfrac{1}{2}\sum_s{}' p_s''^2 + \dfrac{1}{2}\sum_{r,s}{}' c_{r,s}\xi_r''\xi_s'' \\ H_1 = \dfrac{1}{2\rho_2}\left\{m - \sum_s{}' (\xi_s' p_s'' - \xi_s'' p_s')\right\}^2 \end{cases} \tag{2. 29}$$

Our purpose is to solve the eigen-value problem for the operator $H' + H'' + H_1$. To do this we notice that, so far as H' and H'' are concerned, they are quadratic in coordinates and momenta. We may expect, therefore, that we can solve our problem by using the normal coordinates. As to H' it is easy to find out the normal coordinates: we first introduce $\xi'(\mathfrak{k})$ and $p'(\mathfrak{k})$ by the inverse transformation of (2. 3):

$$\xi_s' = \int \xi'(\mathfrak{k})\varphi_s(\mathfrak{k})d\mathfrak{k},$$

$$p_s' = \int p'(\mathfrak{k})\varphi_s(\mathfrak{k})d\mathfrak{k}, \quad s=0,\ 1,\ 2,\ \ldots\ldots \tag{2. 30}$$

Then in terms of this new canonical variables H' has the form:

$$H' = \frac{1}{2}\int\left\{p'(\mathfrak{k})^2 + K^2\xi'(\mathfrak{k})^2\right\}d\mathfrak{k} - l\int g(k)\sqrt{K}\ \xi'(\mathfrak{k})d\mathfrak{k}. \tag{2. 31}$$

The potential energy for the coordinate $\xi'(\mathfrak{k})$ in the Hamiltonian (2. 31) is

$$\text{Pot. energy} = \frac{1}{2}K^2\xi'(\mathfrak{k})^2 - lg(k)\sqrt{K}\ \xi'(\mathfrak{k}), \tag{2. 32}$$

and it contains the term linear in $\xi'(\mathfrak{k})$, so that the position of the equilibrium for the "$\mathfrak{k}$-th oscillator" is not the position $\xi'(\mathfrak{k})=0$ but it is shifted from there by the distance

$$\xi'(\mathfrak{k})_{eq} = lg(k)/K\sqrt{K} \qquad (2.\ 33)$$

This shift of the equilibrium position gives the mean value of the $\mathfrak{k}$-th component of the field due to the existence of the nuclear particle: $\xi'(\mathfrak{k})_{eq}$ gives the $\mathfrak{k}$-th component of the self-field.

The linear term in (2. 31) can be removed, if we displace the coordinate origin for $\xi'(\mathfrak{k})$ to this position of equilibrium. Thus we introduce the canonical transformation from $\xi'(\mathfrak{k})$, $p'(\mathfrak{k})$ to $\bar{\xi}'(\mathfrak{k})$, $\bar{p}'(\mathfrak{k})$ by means of

$$\begin{cases} \bar{\xi}'(\mathfrak{k}) = \xi'(\mathfrak{k}) - l\dfrac{g(k)}{K\sqrt{K}} \\ \bar{p}'(\mathfrak{k}) = p'(\mathfrak{k}). \end{cases} \qquad (2.\ 34)$$

In terms of these barred variables H' is given by

$$H' = \frac{1}{2}\int\left\{\bar{p}'(\mathfrak{k})^2 + K^2\bar{\xi}'(\mathfrak{k})^2\right\}d\mathfrak{k} - \frac{l^2}{2}\boldsymbol{K}_1^3, \qquad (2.\ 35)$$

thus H' has been brought into the normal form.

Before entering to the transformation of H'' into the normal form, it is necessary to see how H_1 is affected by the displacement of origin mentioned above. We introduce thus (2, 33) into H_1. Noting

$$\xi_s' = \int\xi'(\mathfrak{k})\varphi_s(\mathfrak{k})d\mathfrak{k} = \int\bar{\xi}'(\mathfrak{k})\varphi_s(\mathfrak{k})d\mathfrak{k} + l\int\frac{g(k)}{K\sqrt{K}}\varphi_s(\mathfrak{k})d\mathfrak{k},$$

we put

$$\begin{cases} \displaystyle\int\bar{\xi}'(\mathfrak{k})\varphi_s(\mathfrak{k})d\mathfrak{k} = \bar{\xi}_s \\ \displaystyle\int\frac{g(k)}{K\sqrt{K}}\varphi_s(\mathfrak{k})d\mathfrak{k} = Y_s, \quad Y_0 = \frac{\boldsymbol{K}_1^3}{\sqrt{\boldsymbol{K}_{-1}^5}}, \end{cases} \qquad (2.\ 36$$

then we obtain

$$\xi_s' = \bar{\xi}_s' + lY_s \qquad (2.\ 37)$$

and H_1 is expressed now in terms of the barred variables:

$$H_1 = \frac{1}{2}\,\frac{1}{(lY_0 + \bar{\xi}_0')^2}\left\{m - l\sum_s{}' Y_s p_s'' - \sum_s{}'(\bar{\xi}_s' p_s'' - \xi_s''\bar{p}_s')\right\}^2. \qquad (2.\ 38)$$

Since the zero-point amplitude of ξ_0 will be of the order of $l/\boldsymbol{K}$, the first term lY_0 in the denominator of (2. 38) is very large compared with the second term $\bar{\xi}_0'$. We can, therefore, expand the factor $1/(lY_0+\bar{\xi}_0')^2$ in (2. 38) into a power series of $1/l$. Expanding also the numerator of (2, 38), we obtain

$$H_1=\underbrace{\frac{1}{2Y_0^2}\Bigl(\sum_s{}' Y_s p_s''\Bigr)^2}_{\text{term of } l_0}\underbrace{-\frac{m}{lY_0^2}\sum_s{}' Y_s p_s''+\ldots\ldots}_{\text{terms of } l^1}+\underbrace{\frac{m^2}{2l^2Y_0^2}+\ldots\ldots}_{\text{terms of } l^2}\underbrace{+\ldots\ldots}_{\text{higher terms}} \qquad (2.\ 39)$$

We have written in (2, 39) the term of l^0 completely, but among terms of l^{-1} only the term proportional to m and among terms of l^{-2} only the term proportional to m^2. All these terms written in (2. 39) contain no singlly dashed variables and are quadratic in doublly dashed variables. Therefore, these terms added to H'' can be treated by introducing the normal-coordinates. The terms which are not written in (2. 39) are, on the other hand, products of both singlly dashed and doublly dashed variables and are higher degree than quadratic. These terms can not be, therefore, dealt with by introducing the normal coordinates. But these terms are all small of the order of $1/l$, so that, if needed, their effect can be accounted for afterwards by means of successive approximation.

We denote now these small terms by R, then we have

$$H''+H_1=$$
$$\frac{1}{2}\sum_s{}' p_s''^2+\frac{1}{2}\sum_{r,s}{}' c_r\,\xi_r''\xi_s''+\frac{1}{2Y_0}\Bigl(\sum_s{}' Y_s p_s''\Bigr)^2-\frac{m}{lY_0^2}\sum_s{}' Y_s p_s''$$
$$+\frac{m^2}{2l^2Y_0^2}+R \qquad (2.\ 41)$$

In order to eliminate the term $-\frac{m}{lY_0^2}\sum_s{}' Y_s p_s''$ which is linear in p'', we displace here the origin of p'' to its position of equilibrium by means of the canonical transformation

$$\begin{cases} \bar{p}_s''=p_s''-\dfrac{m}{l\mathbf{K}_s}Y_s \\ \bar{\xi}''=\xi_s''. \end{cases} \qquad (2.\ 41)$$

Then $H''+H_1$ is reduced in to

$$\begin{cases} H'' + H_1 = H_0'' + R \\ \quad H_0'' = \frac{1}{2} \sum_s{}' \bar{p}_s''^2 + \frac{1}{2} \sum_{r,s}{}' c_{r,s} \bar{\xi}_r'' \bar{\xi}_s'' + \frac{1}{2l Y_0} (\sum_s{}' Y_s p_s'')^2 + \frac{m^2}{2l^2 \boldsymbol{K}_3} \end{cases} \tag{2. 42}$$

which contains no more the linear term. The term R in (2, 42) is the small quantity of the order of $1/l$ and does not contain m. The shift of the equilibrium position of p_s'' owing to the linear term:

$$p_{s,eq}'' = \frac{m}{l \boldsymbol{K}_3} Y_s \tag{2. 43}$$

gives the mean value of the s-th component of the self-field.

The total Hamiltonian is now of the form

$$\begin{aligned} H &= H' + H_0'' + R \\ &= -\frac{l}{2} \boldsymbol{K}_1^3 + \frac{m^3}{2l^2 \boldsymbol{K}_3} + \frac{1}{2} \int \left\{ \bar{p}_s'(\mathfrak{k})^2 + K^2 \bar{\xi}'(\mathfrak{k})^2 \right\} d\mathfrak{k} \\ &\quad + \frac{1}{2} \sum{}' \bar{p}_s''^2 + \frac{1}{2} \sum_{r,s}{}' c_{r,s} \bar{\xi}_r'' \bar{\xi}_s'' + \frac{1}{2 Y_0} (\sum{}' Y_s \bar{p}_s'')^2 + R. \end{aligned} \tag{2. 44}$$

In this expression H' has already the normal form, and H_0'' is a quadratic form which can be brought into the normal form.

Our next task is thus to find the normal coordinates for H_0''. But before entering to this problem, we shall notice that, neglecting the zero-point energy, (2. 44) gives

$$K_0 = -\frac{l}{2} \boldsymbol{K}_1^3 + \frac{m^2}{2l^2 \boldsymbol{K}_3} \tag{2. 45}$$

for the energies of the lowest states of the system. We can see further that the first correction due to the term R vanishes and the second correction does not contain m. The dependence of (2. 25) on m is, therefore, correct up to the order of l^{-2}.

We now see that (2. 45) agrees perfectly with the classical result (1. 33). In this way our expection that the classical treatment will give the correct result is justified.

We can also calculate the mean value of the self-field. As mentioned before, the shifts of equlibrium positions of the coordinates and the momenta give the mean values for the corresponding expansion coefficients of the self-field. In fact, it can be shown that (2. 33) and (2. 43) give rise to the mean field which agrees in case of large l with the classical self-field (1. 32).

In this way the first term of Wentzel's solution gives the result agreeing perfectly with the classical result. In the symmetrical theory, however, such an agreement is obtained when we first improve the classical result by replacing m^2 by $J(J+1)$.

Such an agreement between classical and quantum-theoretical results is obtained also in the scattering problem as we shall show in the following.

C:—Scattering of a mesotron.

If we neglect the small term R, the variables $\xi'(\mathfrak{k})$ are already the normal coordinates for the total Hamiltonian (2. 44). This means that for the singlly dashed vibrations of the mesotron field the plane waves are already the normal modes. We obtain thus the result: in the approximation in which the small term R is neglected the mesotrons polarized in the direction of $\boldsymbol{\xi}_0$ (we shall call these mesotrons "radial mesotrons") are not scattered by the nuclear particle.

In order to study the scattering of the mesotrons polarized in the direction perpendicular to $\boldsymbol{\xi}_0$ (we shall call these mesotrons "tangential mesotrons") we must find out the normal modes of vibration for the degree of freedom with doublly dashed coordinates. Thus we require the normal coordinates for H_0''.

The process of obtaining the normal coordinates is a purely classical one independent of quantization. So we shall regard, for a while, $\bar{\xi}_s''$ and $\bar{p}_s''$ appearing in H_0'' as ordinary c-numbers. Further, as it is more convenient to represent the normal modes of vibration in the momentum representation in studying the scattering, we introduce the canonical variables $\alpha(\mathfrak{k})$ and $\beta(\mathfrak{k})$ by means of

$$\begin{cases} \bar{\xi}_s'' = \int \alpha(\mathfrak{k}) \varphi_s(\mathfrak{k}) d\mathfrak{k}, \\ p_s'' = \int \beta(\mathfrak{k}) \varphi_s(\mathfrak{k}) d\mathfrak{k}. \end{cases} \tag{2. 46}$$

In terms of these variables H_0'' is found to be

$$H_0''=\frac{1}{2}\int\beta(\mathfrak{k})^2d\mathfrak{k}-\frac{\boldsymbol{K}_{-1}^5}{\boldsymbol{K}_1^3}\int\varphi_0(\mathfrak{k})\beta(\mathfrak{k})d\mathfrak{k}\int\frac{\varphi_0(\mathfrak{k})}{K^2}\beta(\mathfrak{k})d\mathfrak{k}$$
$$+\frac{\boldsymbol{K}_{-1}^{10}}{2\boldsymbol{K}_1^6}\left\{\int\frac{\varphi_0(\mathfrak{k})}{K^2}\beta(\mathfrak{k})\,d\mathfrak{k}\right\}^2+\frac{1}{2}\int\alpha(\mathfrak{k})^2K^2d\mathfrak{k}$$
$$-\int\varphi_0(\mathfrak{k})K^2\alpha(\mathfrak{k})d\mathfrak{k}\int\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k}+\frac{\boldsymbol{K}_{-3}^7}{2\boldsymbol{K}_{-1}^5}\left\{\int\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k}\right\}^2, \qquad (2.\ 47)$$

where $\alpha(\mathfrak{k})$ and $\beta(\mathfrak{k})$ are thought as a set of coordinates and momenta which are ordinary c-numbers. The equation of motion for $\alpha(\mathfrak{k})$ is then obtained:

$$\ddot{\alpha}(\mathfrak{k})=-K^2\alpha(\mathfrak{k})+\left\{K^2-\frac{\boldsymbol{K}_{-3}^7}{\boldsymbol{K}_1^3}\frac{1}{K^2}\right\}\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k}$$
$$+\frac{\boldsymbol{K}_{-1}^5}{\boldsymbol{K}_1^3}\frac{1}{K^2}\varphi_0(\mathfrak{k})\int K^2\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k}. \qquad (2.\ 48)$$

As is evident from the fact that (2. 44) does not contain $\bar{\xi}_s''$ and $\bar{p}_s''$ with $s=0$, there must be one normal vibration having the wave form $\varphi_0(\mathfrak{k})$ and the vanishing frequency. In fact, it can be verified by inserting $\alpha(\mathfrak{k})=\varphi_0(\mathfrak{k})$ into (2. 48). Because the vibration of this wave form should be abcent for the tangential mesotron, our normal modes must be orthogonal to this singular mode.

This fact simplifies our problem. We may, namely, use the simpler equation

$$\ddot{\alpha}(\mathfrak{k})=-K^2\alpha(\mathfrak{k})+\frac{\boldsymbol{K}_{-1}^5}{\boldsymbol{K}_1^3}\frac{1}{K^2}\varphi_0(\mathfrak{k})\int K^2\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k} \qquad (2.\ 49)$$

in place of (2. 28), because we may put $\int\varphi_0(\mathfrak{k})\alpha(\mathfrak{k})d\mathfrak{k}=0$ by the reason mentioned.

Let the wave form of the normal modes for the vibration satisfying the equation of motion (2. 49) be denoted by $\psi(\mathfrak{k})$. Then, this $\psi(\mathfrak{k})$ is obtained by solving the integral equation

$$(K^2-P^2)\psi(\mathfrak{k})=\frac{\boldsymbol{K}_{-1}^5}{\boldsymbol{K}_1^3}\frac{1}{K^2}\varphi_0(\mathfrak{k})\int K^2\varphi_0(\mathfrak{k})\psi(\mathfrak{k})d\mathfrak{k}, \qquad (2.\ 50)$$

thereby the eigen-value P^2 gives the square of the frequency for this oscillation. Solving now (2. 50), imposing the boundary condition saying that there exist at infinity only the incident plane wave with the propagation vector $\mathfrak{p}$ and the outgoing spherical wave scattered by the nuclear particle, we obtain

$$\psi_{\mathfrak{p}}(\mathfrak{k})=\left[\delta(\mathfrak{k}-\mathfrak{p})-\frac{\boldsymbol{K}_{-1}^{5}\varphi_0(\mathfrak{p})}{\boldsymbol{K}_3+(2\pi)^2 i\dfrac{\varphi_0(\mathfrak{p})p}{2}}\frac{\varphi_0(\mathfrak{k})}{K^2(K+P)}\right.$$

$$\left.\left\{\frac{1}{K-P}+i\pi\delta(K-P)\right\}\right]e^{-iPt}, \tag{2. 51}$$

where

$$P^2=p^2+x^2,\quad p=|\mathfrak{p}|. \tag{2. 52}$$

In this calculation we have neglected the terms of higher powers of $P/\boldsymbol{K}$ assuming $P\ll\boldsymbol{K}$. It is easily verified that $\psi(\mathfrak{k})$ thus obtained is in fact orthogonal to φ_0.

From (2. 51) we obtain the scattering cross-section in the way described in § 1, C. Using (2. 2) for $\varphi_0(\mathfrak{k})$ we obtain then

$$\sigma''=\frac{\pi}{p^2}\frac{(4\pi^2 g(p)^2 Pp)^2}{\boldsymbol{K}_3^2 P^4+\frac{1}{4}(4\pi^2 g(p)^2 Pp)^2}. \tag{2. 53}$$

We have written here σ'' with double dashes because this cross-section is of the tangential mesotrons. We have stated above that the radial mesotrons are not scattered by the nuclear particle, so that we may write

$$\sigma'=0. \tag{2. 54}$$

We assume now that the incident mesotrons is a positive one. This state of positive charge can be regarded as a superposition of the state polarized tangentially and the state polarized radially with an equal weight. Then we see that just a half of the incident positive mesotrons will be scattered, and the scattered mesotrons will be polarized tangentially. Now, the state of tangential polarization can be regarded as a superposition of the state of positive charge and the state of negative charge with an equal weight, so that a half of the scattered mesotrons will have positive charge and the other half negative. Thus the probability for the incident positive mesotrons to be scattered keeping their charge is equal to the probability for them to be scattered changing their charge, and the cross-section for each process will de given by 1/4 of (2, 53):

$$\sigma_{+}=\sigma_{-}=\frac{1}{4}\frac{\pi}{p^2}\frac{(4\pi^2 g(p)^2 Pp)^2}{\boldsymbol{K}_3^2 P^4+\frac{1}{4}(4\pi^2 g(p)^2 Pp)^2} \tag{2. 55}$$

If we compare (2. 55) with our classical cross-section (1. 47), we find the perfect agreement. As stated before, the cross-section obtained is considerablly smaller than that obtained by the perturbation theory, and does never exceed the square of the wave length of the incident mesotron. Further this cross-section vanishes when the kinetic energy of the incident mesotron tend to zero, so that it is very small for slow mesotrons.

It is here to be noted that Oppenheimer and Schwinger have calculated the scattering cross-section for the pseudoscalar mesotrons applying Wentzel's method. They have found the cross-section of the same form as (2. 55). They have further interpreted the fact that it vanishes with vanishing kinetic energy of the incident mesotron as due to the situation that in case of pseudoscalar mesotron P-wave is scattered. But our result shows that also in the case of longitudinal vector mesotron in which S-wave is scattered the cross-section shows the same behavior. Wentzel has found, on the other hand, that in case of scalar mesotron, the cross-section does not vanish with vanishing velocity of the incident mesotron*. The ground for the different behavior of the cress-section in the pseudoscalar as well as the longitudinal vector theory on one hand and in the scalar theory on the other hand is thus to be found, not in the difference between S- and P-scattering, but in that the singularity of the self-field is in the case of scalar theory not strong enough and hence the inertia of the self-field is insufficient to reduce the scattering.

D :—Limit of applicability of Wentzel's method.

As stated before, the method of Wentzel is based on the possibility of separation of the three kinds of motion. According to the statement in § 2, A, the condition for the separability of the inner spin motion from the vibrational motion is

$$l^2 K^2 \gg 1, \qquad (2.\ 56)$$

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* All formulas obtained in this paper can be applied also to the case of scalar mesotrons, if one use

$$g(k) = \frac{F(k)}{K} \quad \text{and} \quad l = \frac{1}{2\pi} \frac{g}{\sqrt{hc}}$$

instead of (1, 6) and (1, 11). In this way we can conclude that σ_+ and σ_- do not vanish for vanishing velocity of the incident mesotron.

and the condition for the separability of the vibrational from the rotational motion is

$$l^2\boldsymbol{K}^2=\frac{\boldsymbol{K}}{\varkappa}. \tag{2. 57}$$

Because we assume $\boldsymbol{K}\gg\varkappa$, the latter condition requires stronger coupling than the former.

Comparing (2. 57) with (2. 56), we see that (2. 57) requires the smallness of the interval between rotational levels as compared with $\varkappa$. Thus the condition (2. 57) is the same as the condition for the existence of the stable proton-isobars. Wentzel's method is thus applicable when and only when the coupling is so strong that stable proton-isobars are possible.

We may consider, however, that this method will give sufficiently reliable results also when the weaker condition

$$\frac{\boldsymbol{K}}{\varkappa}> l^2\boldsymbol{K}^2\gg 1 \tag{2. 58}$$

is satisfied, because, in such a case, among many vibrational degrees of freedom many are still separable from the rotational motion. In such a case proton-isobars are not stable in its true sence, but are metastable and will have considerablly long life.

This situation will be explained more precisely: Let us imagine that the proton-isobar with charge 2 is metastable in the sence mentioned above. As this state is not stable, this proton-isobar will dissociate into ordinary proton and a positive mesotron (or eventually several positive and negative mesotrons). Now, in the initial state of the system, i.e. in the state of charge 2 and no free mesotrons, all normal vibrations of the field are in the ground states. The total angular momentum 3/2 belongs in this state to the rotation of the system as a whole, so that this angular momentum is so to speak shared to all field oscillators. In the dissociated state, on the other hand, in which one ordinary proton and one positive mesotron are existing, one field oscillator is excited to the state of angular momentum 1, so that only remaining angular momentum 1/2 is shared to all oscillators. Thus, in order that a proton-isobar with charge 2 dissociates into an ordinary proton and a positive mesotron, one unit of angular momentum of the total 3/2 units, which has been before dissociation shared to all oscillators, must concentrate to one oscillator. Since there are many oscillators in our system* and majority of them are still uncoupled to the rotational motion, a considerablly long time wiil be required for this concentration of the angular momentum.

The existence of such metastable proton-isobars will be implied also by the calculation of the next paragraph based on a quite different foundation. This latter calculation will also suggest that such proton-isobars will decay not only into one definite final particle emitting a single mesotron but there are many possible final particles and, at the same time, several mesotrons will be emitted in these transformations. (To be continued.)

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* The number of oscillators is of the order of $(\boldsymbol{K}/x)^3$.

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Quantum Mechanics of Circularly Polarized Photons.
I. General Theory.

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Introduction and summary.

The quantum mechanics of light was worked out first by Dirac⁽¹⁾. The method of quantum-mechanical description of light has been improved, and many problems on emissions, absorptions, scatterings, dispersions etc. of light have been treated in succession by many authors. However, the phenomena involving circularly polarized photons have not yet been fully discussed, so far as the present author is aware of.

The most familiar one of such phenomena in an elementary process may be the emission of spectral components in the direction of a magnetic field in the case of the normal Zeeman effect. The polarization of these components has not yet been explained on the basis of the quantum mechanics. Its consequent explanation has been ignored since Lorentz gave it by his electron theory which loses now its general validity.

The purpose of this paper is to work out the quantum mechanics of the circularly polarized light in the form convenient to a practical use. For this purpose the quantum mechanics of a system of photons with definite momenta is summarized in the form independent of their polarizations in the first two sections. The Hamiltonian, the momentum and the spin angular momentum of the system are given by the equations (2. 4). The former two have their eigenvalues in this basic states of a second quantization.

We next discuss the transformation of polarization vectors of photons. The transformation from a state of linear polarization to a state in which the spin of the system has also its eigenvalue is determined. We find that

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(1) P. A. M. Dirac, Proc. Roy. Soc. London (A), 114 (1927), 243.

such a transformation is unique except a trivial transformation such as (3. 20). It is essentially given by (3. 16) and (3. 17). The energy, the momentum and the spin of the system are transformed into those given by (3. 21).

In the fourth section a system of circularly polarized photons is discussed. We find that the spin has also its eigenvalue in this system. We have seen in the previous section that such a polarization state of photons is unique. The basic states of the second quantization in which the spin of the system has its eigenvalue are therefore those of an assemblage of circularly polarized photons, and vice versa. In the fifth section the eigenfunction of a system of circularly polarized photons is determined.

In the second part the interaction between photons and an atom or a molecule is represented in the form ready to use in the case of any polarization, and the general theory is applied to the polarization of spectral components emitted by an atom in the direction of an applied magnetic field in the case of the normal Zeeman effect. The complete agreement between our theory and the experiment regarding their polarization is verified.

1. Canonical description of light as a system of plane waves.

According to the Maxwell equation, the electromagnetic field ($\boldsymbol{E}$, $\boldsymbol{H}$) of a light wave is solenoidal, and it can be described by a solenoidal vector potential $\boldsymbol{A}$ as follows:

$$\boldsymbol{E}=-\frac{1}{c}\dot{\boldsymbol{A}}, \quad \boldsymbol{H}=\mathrm{rot}\,\boldsymbol{A}. \tag{1. 1}$$

The energy, the momentum and the angular momentum of the field are given respectively by

$$\left.\begin{aligned} H&=\frac{1}{8\pi}\int(\boldsymbol{E}^2+\boldsymbol{H}^2)dV \\ \boldsymbol{G}&=\frac{1}{4\pi c}\int[\boldsymbol{E}\boldsymbol{H}]dV \\ \boldsymbol{J}&=\frac{1}{4\pi c}\int\Big[x[\boldsymbol{E}\boldsymbol{H}]\Big]dV \end{aligned}\right\} \tag{1. 2}$$

where the field and the integrals are assumed to extend over all space.

This assumption involves that of the convergence of the integrals, consequently the sufficiently strong vanishing of the integrands at infinity.

The last integral can be divided into two parts by partial integrations⁽²⁾ on account of the last mentioned property of the integrands:

$$\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S} \tag{1. 3}$$

where

$$\left.\begin{aligned}\boldsymbol{L}&=\frac{1}{4\pi c}\int\sum_{j=1}^{3}E_j[\boldsymbol{x}\nabla]A_j dV\\ \boldsymbol{S}&=\frac{1}{4\pi c}\int[\boldsymbol{E}\boldsymbol{A}]dV\end{aligned}\right\} \tag{1. 4}$$

This resolution corresponds just to Darwin's⁽³⁾ in a space of propagation vectors of plane waves. The above resolution does not depend on the special representation as Darwin's. $\boldsymbol{L}$ means the quantum-mechanical orbital angular momentum of the field. The reason will be explained in the paper which will shortly be published in this journal. $\boldsymbol{S}$ is therefore the spin angular momentum of the field.

The vector potential satisfies the wave equation:

$$\frac{1}{c^2}\ddot{\boldsymbol{A}}-\triangle\boldsymbol{A}=0 \tag{1. 5}$$

Now we assume the space to be limited in the sufficiently large cube V of edge length $\sqrt[3]{V}$. The general, solenoidal and real solution of this equation is then given by the superposition of transversal plane waves:

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(2) Substituting the rotation of the vector potentiol for the magnetic field in the integrand of the last integral of (1, 2), we have:

$$\boldsymbol{J}=\frac{1}{4\pi c}\int\left\{\sum_{j=1}^{3}E_j[\boldsymbol{x}\triangledown]A_j-[\boldsymbol{x}(\boldsymbol{E}\triangledown)\boldsymbol{A}]\right\}dV$$

Owing to the solenoidal property of $\boldsymbol{E}$, the latter term becomes the form of (1. 4) by a partial integration.

(3) C. G. Darwin, Proc. Roy. Soc. London (A), 136 (1932), 36. It is cited, in W. Heitler, Quantum theory of radiation, (1936) p. 63, that the angular momentum of light is carefully discussed in a paper by N. Bohr, C. Mannebeck, G. Placzek and L. Rosenfeld. To my regret I can not see this paper.

$$\boldsymbol{A}(\boldsymbol{x},t)=c\sqrt{\frac{4\pi}{V}}\sum_{\text{whole }k}\Big\{\boldsymbol{e}_1(\boldsymbol{k})A_1(\boldsymbol{k})\sin(\omega t-\boldsymbol{k}\boldsymbol{x}+\varepsilon_1(\boldsymbol{k}))$$
$$+\boldsymbol{e}_2(\boldsymbol{k})A_2(\boldsymbol{k})\sin(\omega t-\boldsymbol{k}\boldsymbol{x}+\varepsilon_2(\boldsymbol{k})\Big\} \qquad (1.\ 6)$$

where

$$\left.\begin{array}{l} k_j=\dfrac{2\pi}{\sqrt[3]{V}}n_j, \quad \begin{array}{l} j=1,2,3,\ n_j=0,\ \pm1,\ \pm2,\ldots\ldots \\ (n_1=n_2=n_3=0 \text{ is excluded}) \end{array} \\ \omega=c|\boldsymbol{k}| \end{array}\right\} \qquad (1.\ 7)$$

$$\left.\begin{array}{l} \boldsymbol{e}_1(\boldsymbol{k})\boldsymbol{e}_2(\boldsymbol{k})=\boldsymbol{k}\boldsymbol{e}_1(\boldsymbol{k})=\boldsymbol{k}\boldsymbol{e}_2(\boldsymbol{k})=0 \\ |\boldsymbol{e}_1(\boldsymbol{k})|=|\boldsymbol{e}_2(\boldsymbol{k})|=1 \end{array}\right\} \qquad (1.\ 8)$$

$\varepsilon_1(\boldsymbol{k})$ and $\varepsilon_2(\boldsymbol{k})$ are arbitrary phase constants which depend on the propagation vector $\boldsymbol{k}$, and the summation extends over whole lattice points of $\boldsymbol{k}$-space. It is to be noted that the $\boldsymbol{k}$-lattice space does not contain an origin. The quantities contained in the above expressions are all real. Each summand of (1. 6) represents in general a elliptically polarized wave including a linearly or a circularly polarized wave.

If we introduce the variables

$$\boldsymbol{q}_k=\boldsymbol{e}_1(\boldsymbol{k})A_1(\boldsymbol{k})\sin(\omega t+\varepsilon_1)+\boldsymbol{e}_2(\boldsymbol{k})A_2(\boldsymbol{k})\sin(\omega t+\varepsilon_2) \qquad (1.\ 9)$$

they satisfy the equations of two dimensional isotropic harmonic motion:

$$\ddot{\boldsymbol{q}}_k+\omega^2\boldsymbol{q}_k=0 \qquad (1.\ 10)$$

and the Fourier expansion of $\boldsymbol{A}$ is given by(4)

$$\boldsymbol{A}(\boldsymbol{x},t)=c\sqrt{\frac{4\pi}{V}}\sum_{\text{half }k}\Big\{(\boldsymbol{q}_k+\boldsymbol{q}_{-k})\cos\boldsymbol{k}\boldsymbol{x}-\frac{1}{\omega}(\dot{\boldsymbol{q}}_k-\dot{\boldsymbol{q}}_{-k})\sin\boldsymbol{k}\boldsymbol{x}\Big\} \qquad (1.\ 11)$$

where the summation extends over one half of lattice points of the $\boldsymbol{k}$-space, i.e. over one of the two sets of $\boldsymbol{k}$-lattice points which are complementary in the $\boldsymbol{k}$-space and transform mutually by an inversion of the $\boldsymbol{k}$-space.

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(4) It is to be noted that the Fourier coefficients contain $\boldsymbol{q}$ or $\dot{\boldsymbol{q}}$. This point has been misunderstood in W. Heitler, Quantum theory of radiation, (1936) p. 41.

The Parseval identities fos $\boldsymbol{E}$ and $\boldsymbol{H}$ are⁽⁵⁾

$$\left.\begin{aligned} \int_V \boldsymbol{E}^2 dV &= 2\pi \sum_{\text{half } k} \left\{ (\dot{\boldsymbol{q}}_k + \dot{\boldsymbol{q}}_{-k})^2 + \omega^2 (\boldsymbol{q}_k - \boldsymbol{q}_{-k})^2 \right\} \\ \int_V \boldsymbol{H}^2 dV &= 2\pi \sum_{\text{half } k} \left\{ (\dot{\boldsymbol{q}}_k - \dot{\boldsymbol{q}}_{-k})^2 + \omega^2 (\boldsymbol{q}_k + \boldsymbol{q}_{-k})^2 \right\} \end{aligned}\right\} \tag{1. 12}$$

The energy is accordingly given by

$$H = \frac{1}{2} \sum_{\text{whole } k} (\dot{\boldsymbol{q}}_k^2 + \omega^2 \boldsymbol{q}_k^2). \tag{1. 13}$$

Hence, if we introduce a momentum defined by

$$\boldsymbol{p}_k = \dot{\boldsymbol{q}}_k \tag{1. 14}$$

the equation of motion (1. 10) can be written in the canonical form. And we can consider the field as a system of two dimentional isotropic harmonic oscillators, $\boldsymbol{q}_k$ and $\boldsymbol{p}_k$ being their canonically conjugate coordinate and momentum. Each oscillator corresponds to a running plane wave with a definite propagation vector. An oscillator characterized by $\boldsymbol{k}$ will be referred to as $\boldsymbol{k}$-oscillator.

Using these canonical variables of oscillators, $\boldsymbol{A}$, H, $\boldsymbol{G}$ and $\boldsymbol{S}$ can be written as follows:

$$\left.\begin{aligned} \boldsymbol{A} &= c\sqrt{\frac{4\pi}{V}} \sum_{\text{whole } k} \left(\boldsymbol{q}_k \cos \boldsymbol{kx} - \frac{1}{\omega} \boldsymbol{p}_k \sin \boldsymbol{kx}\right) \\ H &= \frac{1}{2} \sum_{\text{whole } k} (\boldsymbol{p}_k^2 + \omega^2 \boldsymbol{q}_k^2) \\ \boldsymbol{G} &= \frac{1}{2} \sum_{\text{whole } k} \frac{\boldsymbol{k}}{\omega} (\boldsymbol{p}_k^2 + \omega^2 \boldsymbol{q}_k^2) \\ \boldsymbol{S} &= \sum_{\text{whole } k} [\boldsymbol{q}_k \boldsymbol{p}_k] \end{aligned}\right\} \tag{1. 15}$$

The last expression is independent of the position of the coordinate

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(5) It is notable that this expression contains the cross-products $q_k q_{-k}$ and $\dot{q}_k \dot{q}_{-k}$. This point has been often overlocked in the standard text-books, e.g. W. Heitler, l.c., G. Wentzel, Geiger-Scheed's Handb. der Phys., 24/1 (1933), 740, W. Pauli, ibid, 247.

origin of the configuration space. It represents a sum of eigen-angular momenta of plane waves.

2. A system of photons with definite momenta.

As is well known, if we consider the system as an assemblage of two dimentional quantum-mechanical oscillators, the eigenvalues of the energy of the system are, owing to the second equation of (1. 15), given by

$$E=\sum_{\text{whole } k} \hbar\omega(N_k+1) \tag{2. 1}$$

where $N_k(\doteqdot 0,\ 1,\ 2,\ \ldots\ldots)$ is the quantum number of the two dimensional $\boldsymbol{k}$-oscillator. This quantum number is interpreted as the number of photons with momenta $\hbar\boldsymbol{k}$ which will be referred to as $\boldsymbol{k}$-photons. $\boldsymbol{p}_k$ and $\boldsymbol{q}_k$ are their canonically conjugate Hermitian vector operators, which satisfy the well known commutation relations.

If we introduce new operators by a reversible transformation

$$\boldsymbol{a}_k=\sqrt{\frac{\omega}{2\hbar}}\left(\boldsymbol{q}_k+\frac{i}{\omega}\boldsymbol{p}_k\right) \tag{2. 2}$$

the Hamiltonian of $\boldsymbol{k}$-oscillator is then written as

$$\frac{1}{2}(\boldsymbol{p}_k^2+\omega^2\boldsymbol{q}_k^2)=\hbar\omega(\boldsymbol{a}_k{}^\dagger\boldsymbol{a}_k+1) \tag{2. 3}$$

where $\dagger$ means the adjoint. These new operators are not Hermitian. The old Hermitian operators $\boldsymbol{q}_k$ and $\boldsymbol{p}_k$ are respectively the Hermitian and skew-Hermitian parts of $\boldsymbol{a}_k$ except their factors. $N_k(=0, 1, 2, 3, \ldots\ldots)$ is the eigenvalue of a Hermitian operator $\boldsymbol{a}_k{}^\dagger\boldsymbol{a}_k$ which represents the number of $\boldsymbol{k}$-photons. The infinite zero-point energy of (2. 1) can be excluded by omitting 1 in the parenthesis on the right-hand side of (2. 3). We adopt this representation of the Hamiltonian throughout the present paper.

The equations (1. 15) are then transformed into

$$\left.\begin{aligned}
\boldsymbol{A}&=\hbar c\sqrt{\frac{2\pi}{V}}\sum_{\text{whole } k}\frac{1}{\sqrt{\hbar\omega}}(\boldsymbol{a}_k e^{i\boldsymbol{kx}}+\boldsymbol{a}_k{}^\dagger e^{-i\boldsymbol{kx}})\\
H&=\sum_{\text{whole } k}\hbar\omega\boldsymbol{a}_k{}^\dagger\boldsymbol{a}_k\\
\boldsymbol{G}&=\sum_{\text{whole } k}\hbar\boldsymbol{k}(\boldsymbol{a}_k{}^\dagger\boldsymbol{a}_k)\\
\boldsymbol{S}&=\sum_{\text{whole } k}\frac{\hbar}{i}[\boldsymbol{a}_k{}^\dagger\boldsymbol{a}_k]
\end{aligned}\right\} \tag{2. 4}$$

We see that the energy and the momentum have their eigenvalues in this second quantization with the basic eigenstates of photons with definite momenta (with arbitrary polarizations): a $\boldsymbol{k}$-photon has a energy $\hbar\omega$ and a momentum $\hbar\boldsymbol{k}$.

3. Transformation of polarization-vectors.

The results of the preceding sections are independent of polarization states of photons. In this section we shall consider the polarization states of photons. As we shall confine ourselves to photons with a definite propagation vector $\boldsymbol{k}$, we shall omit suffices $\boldsymbol{k}$'s from all expressions, until there will be a need of them.

Consider an unitary transformation S of two basic unit vectors of $\boldsymbol{k}$-photons in a plane perpendicular to its propagation vector $\boldsymbol{k}$:

$$\left.\begin{aligned} \boldsymbol{e}_1' &= \boldsymbol{e}_1 S_{12} + \boldsymbol{e}_2 S_{21} \\ \boldsymbol{e}_2' &= \boldsymbol{e}_1 S_{12} + \boldsymbol{e}_2 S_{22} \end{aligned}\right\} \tag{3. 1}$$

$$S^\dagger S = SS^\dagger = 1 \tag{3. 2}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the real unit vectors defined by (1, 8). $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$ are then complex in general and orthonormal in the sense of the unitary geometry. The coefficients of the transformation are given by Hermitian inner products (not by ordinary scalar products):

$$S_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j') \quad i, j = 1, 2. \tag{3. 3}$$

If we expand $\boldsymbol{a}$ by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ or by $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$:

$$\boldsymbol{a} = \boldsymbol{e}_1 a_1 + \boldsymbol{e}_2 a_2 = \boldsymbol{e}_1' a_1' + \boldsymbol{e}_2' a_2' \tag{3. 4}$$

the transformation of the coefficients of these expansions is contragredient to that of the base:

$$\left.\begin{aligned} a_1 &= S_{11} a_1' + S_{12} a_2' \\ a_2 &= S_{21} a_1' + S_{22} a_2' \end{aligned}\right\} \tag{3. 5}$$

The commutation relations for a_1 and a_2 follow from those for $\boldsymbol{q}$ and $\boldsymbol{p}$:

$$a_1a_1^\dagger - a_1^\dagger a_1 = a_2a_2^\dagger - a_2^\dagger a_2 = 1 \tag{3. 6}$$

all other pairs being commutative. The eigenvalues of the Hermitian operators $a_1^\dagger a_1$ and $a_2^\dagger a_2$ represent the numbers of linearly polarized photons with the planes of oscillation respectively parallel to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ [see (1. 1) and (2. 4)]. On account of the unitary property of the matrix S, the above mentioned commutation relations are invariant for the transformation. Further, the number of $\boldsymbol{k}$-photons is invariant also, for it is represented by an invariant operator

$$\boldsymbol{a}^\dagger\boldsymbol{a} = a_1^\dagger a_1 + a_2^\dagger a_2 = a_1'^\dagger a_1' + a_2'^\dagger a_2'. \tag{3. 7}$$

The energy and the momentum are therefore invariant.

We define the canonically conjugate Hermitian operators q_j and p_j of $\boldsymbol{k}$-oscillator with j-polarization as the Hermitian part of $\sqrt{\frac{2\hbar}{\omega}}\,a_j$ and the skew-Hermitian part of $\sqrt{2\hbar\omega}\,a_j$ respectively ($j=1, 2$):

$$\left.\begin{aligned} q_j &= \sqrt{\frac{\hbar}{2\omega}}\,(a_j^\dagger + a_j) \\ p_j &= \sqrt{\frac{\hbar\omega}{2}}\,i(a_j^\dagger - a_j) \end{aligned}\right\}\ j=1, 2 \tag{3. 8}$$

The commutation relations of these operators are obtained from the commutation relations (3. 6) for a_1 and a_2 as follows:

$$q_1p_1 - p_1q_1 = q_2p_2 - p_2q_2 = i\hbar \tag{3. 9}$$

all other pairs being commutative. As the commutation relations (3. 6) for a_1 and a_2 are invariant, (3. 9) is also invariant.

It is to be noted that q_1', q_2', p_1' and p_2' are, in general, not equal to the coefficients, $(\boldsymbol{e}_1', \boldsymbol{q})$, $(\boldsymbol{e}_2', \boldsymbol{q})$, $(\boldsymbol{e}_1', \boldsymbol{p})$ and $(\boldsymbol{e}_2', \boldsymbol{p})$, of expansions of $\boldsymbol{q}$ and $\boldsymbol{p}$ by the transformed base $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$. The transformation of former two or latter two of these coefficients is cogredient to that of a_1 and a_2. The Hermitian properties of the operators can not be reserved and the commutation relations (3. 9) are no more invariant for this transformation of coefficient in general. The transformation of the operators q_1, q_2, p_1 and p_2 defined by (3. 8) is, in contrast with these coefficients, given by an overmatrix derived from S:

$$(q_1' q_2' p_1' p_2') = (q_1 q_2 p_1 p_2) \begin{pmatrix} R & -\omega T \\ \frac{T}{\omega} & R \end{pmatrix} \tag{3. 10}$$

where R and T are respectively the real and the imaginary parts of the matrix S (but not the Hermitian and the skew Hermitian parts).

The most general unitary transformation is given by

$$S = \begin{pmatrix} e^{i\theta_1} \cos \varphi & e^{i(\theta_1 - \theta)} \sin \varphi \\ -e^{i(\theta_2 - \theta)} \sin \varphi & e^{i\theta_2} \cos \varphi \end{pmatrix} \tag{3. 11}$$

where θ_1, θ_2, θ, φ are the real independent parameters of the transformation. If we write the spin angular momentum of $\boldsymbol{k}$-photons as

$$\frac{\hbar}{i} [\boldsymbol{a}^\dagger \boldsymbol{a}] = [\boldsymbol{e}_1 \boldsymbol{e}_2] \hbar \wedge \tag{3. 12}$$

the operator $\wedge$ is Hermitian, and its representation is given by the transformed operators as follows:

$$\left.\begin{aligned} \wedge = \frac{1}{i} (a_1'^\dagger a_2'^\dagger - a_2'^\dagger a_1') & \left\{ \sin^2 \varphi \cos (\theta_1 - \theta_2 + 2\theta) + \cos^2 \varphi \cos (\theta_1 - \theta_2) \right\} \\ + (a_1'^\dagger a_2' + a_2'^\dagger a_1') & \left\{ \sin^2 \varphi \sin (\theta_1 - \theta_2 + 2\theta) - \cos^2 \varphi \sin (\theta_1 - \theta_2) \right\} \\ + (a_1'^\dagger a_1' - a_2'^\dagger a_2') & \sin 2\varphi \sin (\theta_1 - \theta_2 + \theta) \end{aligned}\right\} \tag{3. 13}$$

In order that the representation of $\wedge$ is invariant, the factor of the first term of (3. 13) must be equal to unity and the factors of the second and the third terms must vanish, for $\theta_1 = \theta_2 = \theta = \varphi = 0$ corresponds to the identical transformation. We can easily find that the transformation satisfying these conditions is

$$S = e^{i\theta} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \tag{3. 14}$$

This means that the transformation is essentially the rotation by an angle φ of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, because the trivial transformation $S = e^{i\theta}$ is inessential. In other words, the representation of the spin angular momentum given by $[\boldsymbol{e}_1 \boldsymbol{e}_2] \frac{\hbar}{i} (a_1^\dagger a_2 - a_2^\dagger a_1)$ is characteristic for the basic states of linear polarization.

We shall next seek for a transformation from a base of linear polarization to the base in which the spin angular momentum of the system is represented by operators having eigenvalues. The condition for this requirement is that the factors of the first two terms of (3. 13) vanish and the factor of the last term is equal to unity. This gives

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta_1} & e^{i\theta_2} \\ ie^{i\theta_1} & -ie^{i\theta_2} \end{pmatrix} \tag{3. 15}$$

A product of (3. 14) and (3. 15), in which the latter is multiplied from right, is also a required transformation, but this is neither more or less than (3. 15). We find in fact from the result of carrying out this multiplication that this product can be obtained by simply replacing $\theta_1+\theta+\varphi$ and $\theta_2+\theta-\varphi$ instead of θ_1 and θ_2 of (3. 15) respectively.

As the trivial transformation given by (3. 20) below is inessential, we can put $\theta_1=\theta_2=0$ in (3. 15). In this case we write a_l and a_r in place of a_1' and a_2', and further $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$ in place of $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$. We have then

$$\left.\begin{aligned} a_l&=\frac{1}{\sqrt{2}}(a_1-ia_2) \\ a_r&=\frac{1}{\sqrt{2}}(a_1+ia_2) \end{aligned}\right\} \tag{3. 16}$$

and

$$\left.\begin{aligned} \boldsymbol{e}^+&=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1+i\boldsymbol{e}_2) \\ \boldsymbol{e}^-&=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1-i\boldsymbol{e}_2) \end{aligned}\right\} \tag{3. 17}$$

The representations of $\boldsymbol{a}$ and $\wedge$ become

$$\boldsymbol{a}=\boldsymbol{e}^+a_l+\boldsymbol{e}^-a_r \tag{3. 18}$$

$$\wedge=a_l^\dagger a_l-a_r^\dagger a_r \tag{3. 19}$$

The transformation which reserves this representation of $\wedge$ invariant is

$$S=\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \tag{3. 20}$$

which is inessential. We see that the required transformation is unique except this trivial one, and that it is therefore given essentially by (3. 16) and (3. 17).

The energy, the momentum and the spin of the whole system are now given in terms of a_l and a_r as follows:

$$\left.\begin{aligned} \boldsymbol{H} &= \sum_{\text{whole } \boldsymbol{k}} \hbar\omega\left\{a_l^\dagger(\boldsymbol{k})a_l(\boldsymbol{k}) + a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k})\right\} \\ \boldsymbol{G} &= \sum_{\text{whole } \boldsymbol{k}} \hbar\boldsymbol{k}\left\{a_l^\dagger(\boldsymbol{k})a_l(\boldsymbol{k}) + a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k})\right\} \\ \boldsymbol{S} &= \sum_{\text{whole } \boldsymbol{k}} [\boldsymbol{e}_1(\boldsymbol{k})\boldsymbol{e}_2(\boldsymbol{k})]\hbar\left\{a_l^\dagger(\boldsymbol{k})a_l(\boldsymbol{k}) - a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k})\right\} \end{aligned}\right\} \tag{3. 21}$$

As the commutation relations (3. 6) are invariant, $a_l^\dagger a_l$ and $a_r^\dagger a_r$ have integral eigenvalues N_l and N_r respectively. The energy, the momentum and the spin of the system have their eigenvalues in this representation. Especially the eigenvalue of the spin is.

$$\sum_{\text{whole } \boldsymbol{k}} [\boldsymbol{e}_1(\boldsymbol{k})\boldsymbol{e}_2(\boldsymbol{k})]\hbar\left\{N_l(\boldsymbol{k}) - N_r(\boldsymbol{k})\right\} \tag{3. 22}$$

As will be seen later in the next section, $N_l(\boldsymbol{k})$ and $N_r(\boldsymbol{k})$ represent the numbers of left-circularly and right-circularly polarized $\boldsymbol{k}$-photons respectively, when $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{k}$ are a right-handed system. Thus we see that a left-circularly polarized photon has a spin $+1$ directed to its propagation vector $\boldsymbol{k}$ and a right-circularly polarized photon has a spin -1. We can conversely say, on the basis of the above deduction, that the state of photons having the definite energy, momentum and spin is a circularly polarized one.

Lastly we shall get the canonically conjugate operators of a circularly polarized $\boldsymbol{k}$-oscillator. According to (3. 10) and (3. 17) they are given by

$$\left.\begin{aligned} q_l &= \frac{1}{\sqrt{2}}\left(q_1 + \frac{1}{\omega}p_2\right) \\ p_l &= \frac{1}{\sqrt{2}}(p_1 - \omega q_2) \\ q_r &= \frac{1}{\sqrt{2}}\left(q_1 - \frac{1}{\omega}p_2\right) \\ p_r &= \frac{1}{\sqrt{2}}(p_1 + \omega q_2) \end{aligned}\right\} \tag{3. 23}$$

4. System of circularly polarized photons.

We shall now consider a system of circularly polarized photons and show that the Hermitian operators $a_l{}^\dagger a_l$ and $a_r{}^\dagger a_r$ defined by (3. 16) represent really the numbers of left-circularly and right-circularly polarized photons.

The general solution (1. 6) of the wave equation (1. 5) can also be resolved into the superposition of circularly polarized waves:

$$\boldsymbol{A}(\boldsymbol{x}, t)=c\sqrt{\frac{4\pi}{V}}\sum_{\text{whole } k}\Big[A_l(\boldsymbol{k})\Big\{\boldsymbol{e}_1(\boldsymbol{k})\sin(\omega t-\boldsymbol{kx}+\varepsilon_l(\boldsymbol{k}))$$
$$-\boldsymbol{e}_2(\boldsymbol{k})\cos(\omega t-\boldsymbol{kx}+\varepsilon_l(\boldsymbol{k}))\Big\}+A_r(\boldsymbol{k})\Big\{\boldsymbol{e}_1(\boldsymbol{k})\sin(\omega t-\boldsymbol{kx}+\varepsilon_r(\boldsymbol{k}))$$
$$+\boldsymbol{e}_2(\boldsymbol{k})\cos(\omega t-\boldsymbol{kx}+\varepsilon_r(\boldsymbol{k}))\Big\}\Big] \tag{4. 1}$$

where $\boldsymbol{e}_1(\boldsymbol{k})$ and $\boldsymbol{e}_2(\boldsymbol{k})$ are the real unit vectors defined by (1. 8). In this section we shall assume that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{k}$ form a right-handed system. The terms involving factors with a suffix l then give left-circularly polarized plane waves of light [cf. (1. 1)], and those with a suffix r give right-circularly polarized ones. $\varepsilon_l(\boldsymbol{k})$ and $\varepsilon_r(\boldsymbol{k})$ are phase constants of left-circularly and right-circularly polarized waves respectively. The vector potential given by (1. 6) is the same vector function as that given by (4. 1). These are two representation of the same function by two different sets of basic functions.

If we put

$$\left.\begin{aligned}\boldsymbol{q}_l(\boldsymbol{k})&=A_l\Big\{\boldsymbol{e}_1\sin(\omega t+\varepsilon_l)-\boldsymbol{e}_2\cos(\omega t+\varepsilon_l)\Big\}\\ \boldsymbol{q}_r(\boldsymbol{k})&=A_r\Big\{\boldsymbol{e}_1\sin(\omega t+\varepsilon_r)+\boldsymbol{e}_2\cos(\omega t+\varepsilon_r)\Big\}\end{aligned}\right\} \tag{4. 2}$$

and

$$\boldsymbol{q}_l(\boldsymbol{k})=\boldsymbol{p}_l(\boldsymbol{k}),\quad \boldsymbol{q}_r(\boldsymbol{k})=\boldsymbol{p}_r(\boldsymbol{k}) \tag{4. 3}$$

the canonical variables of a $\boldsymbol{k}$-oscillator defined by (1. 9) and (1. 14) are decomposed as follows:

$$\left.\begin{aligned}\boldsymbol{q}_k&=\boldsymbol{q}_l(\boldsymbol{k})+\boldsymbol{q}_r(\boldsymbol{k})\\ \boldsymbol{p}_k&=\boldsymbol{p}_l(\boldsymbol{k})+\boldsymbol{p}_r(\boldsymbol{k})\end{aligned}\right\} \tag{4. 4}$$

From the definitions (4. 2) and (4. 3) we have

$$\left.\begin{aligned}
&[q_r p_l]+[q_l p_r]=0\\
&p_l p_r+\omega^2 q_l q_r=0\\
&[q_l p_l]=[e_1 e_2]\frac{1}{2\omega}(p_l^2+\omega^2 q_l^2)\\
&[q_r p_r]=-[e_1 e_2]\frac{1}{2\omega}(p_r^2+\omega^2 q_r^2)
\end{aligned}\right\} \tag{4. 5}$$

where suffices k's are omitted from all the quantities for the sake of simplicity.

We substitute the right-hand side of (4. 4) for q_k's and p_k's in (1. 15), and make use of (4. 5). We have then

$$\left.\begin{aligned}
H&=\frac{1}{2}\sum_{\text{whole}\,k}(p_l^2(k)+\omega^2 q_l^2(k)+p_r^2(k)+\omega^2 q_r^2(k))\\
G&=\frac{1}{2}\sum_{\text{whole}\,k}\frac{k}{\omega}(p_l^2(k)+\omega^2 q_l^2(k)+p_r^2(k)+\omega^2 q_r^2(k))\\
S&=\frac{1}{2}\sum_{\text{whole}\,k}\frac{[e_1(k)e_2(k)]}{\omega}\Big\{(p_l^2(k)+\omega^2 q_l^2(k))-(p_r^2(k)+\omega^2 q_r^2(k))\Big\}
\end{aligned}\right\} \tag{4. 6}$$

The position vectors $q_l(k)$ and $q_r(k)$ satisfy an equation of a simple harmonic motion also. We can consider these position vectors as that of two independent oscillators which represent left-circularly and right-circularly polarized plane waves of light respectively.

We now consider the system as an assemblage of quantum-mechanical oscillators representing these circularly polarized lights. As is seen from (4. 6), the energy, the momentum and the spin of the system have their eigenvalues in this case. The basic states of this representation are therefore essentially the same as those characterized by the polarization vectors defined by (3. 17).

Two pairs of the canonically conjugate Hermitian operators of a k-oscillator satisfy the commutation relations

$$q_l p_l - p_l q_l = q_r p_r - p_r q_r = i\hbar \tag{4. 7}$$

all other pairs being commutative. Due to the definitions (4. 2) and (4. 3) these operators can be written in the form

$$\left.\begin{aligned}\mathbf{q}_l&=\frac{1}{\sqrt{2}}\left(\mathbf{e}_1 q_l-\frac{1}{\omega}\mathbf{e}_2 p_l\right)\\ \mathbf{p}_l&=\frac{1}{\sqrt{2}}(\mathbf{e}_1 p_l+\omega\mathbf{e}_2 q_l)\\ \mathbf{q}_r&=\frac{1}{\sqrt{2}}\left(\mathbf{e}_1 q_r+\frac{1}{\omega}\mathbf{e}_2 p_r\right)\\ \mathbf{p}_r&=\frac{1}{\sqrt{2}}(\mathbf{e}_1 p_r-\omega\mathbf{e}_2 q_r)\end{aligned}\right\}\qquad(4.8)$$

for a definite $\mathbf{k}$. The commutation relations between the transformed operators follow from (4. 7):

$$q_l p_l-p_l q_l=q_r p_r-p_r q_r=i\hbar \qquad (4.9)$$

for a definite $\mathbf{k}$, all other pairs being commutative. Further we have

$$\begin{aligned}\mathbf{p}_l^2+\omega^2\mathbf{q}_l^2&=p_l^2+\omega^2 q_l^2\\ \mathbf{p}_r^2+\omega^2\mathbf{q}_r^2&=p_r^2+\omega^2 q_r^2\end{aligned}\qquad(4.10)$$

The transformation (4. 8) is consequently canonical.

If we put

$$\left.\begin{aligned}\mathbf{a}_l&=\sqrt{\frac{\omega}{2\hbar}}\left(\mathbf{q}_l+\frac{i}{\omega}\mathbf{p}_l\right)\\ \mathbf{a}_r&=\sqrt{\frac{\omega}{2\hbar}}\left(\mathbf{q}_r+\frac{i}{\omega}\mathbf{p}_r\right)\end{aligned}\right\}\qquad(4.11)$$

for a definite $\mathbf{k}$, we have by (2. 2) and (4. 4)

$$\left.\begin{aligned}\mathbf{a}_k&=\mathbf{a}_l(\mathbf{k})+\mathbf{a}_r(\mathbf{k})\\ \mathbf{a}_l&=\mathbf{e}^+\sqrt{\frac{\omega}{2\hbar}}\left(q_l+\frac{i}{\omega}p_l\right)\\ \mathbf{a}_r&=\mathbf{e}^-\sqrt{\frac{\omega}{2\hbar}}\left(q_r+\frac{i}{\omega}p_r\right)\end{aligned}\right\}\qquad(4.12)$$

where $\mathbf{e}^+$ and $\mathbf{e}^-$ are the complex vectors defined by (3. 17). It follows from the definition (3. 18) and the result (4. 12) that

$$\left.\begin{aligned} a_l &= \sqrt{\frac{\omega}{2\hbar}}\left(q_l + \frac{i}{\omega}p_l\right) \\ a_r &= \sqrt{\frac{\omega}{2\hbar}}\left(q_r + \frac{i}{\omega}p_r\right) \end{aligned}\right\} \qquad (4.13)$$

where a_l and a_r are the operators defined by (3. 16). Comparing these equations with (3. 8) we see that the operators q_l, p_l, q_r and p_r defined by (4. 8) are just the same as those transformed from a_l and a_r according to (3. 8), namely those given by (3. 23). The operators a_l and a_r defined by (3. 16) are therefore those of left-circularly and right-circularly polarized photons respectively, and the Hermitian operators $a_l^\dagger(\boldsymbol{k})a_l(\boldsymbol{k})$ and $a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k})$ represent the numbers of those photons. This conclusion is just our required answer to be proved.

5. Eigenfunction of a system of photons with a definite spin.

Consider a system of photons with the definite momentum and spin. We shall determine the eigenfunction of this system in a q-representation of linearly polarized state:

$$\left.\begin{aligned} \boldsymbol{q} &= \boldsymbol{e}_1 q_1 + \boldsymbol{e}_2 q_2 \\ \boldsymbol{p} &= \boldsymbol{e}_1 \frac{\hbar}{i}\frac{\partial}{\partial q_1} + \boldsymbol{e}_2 \frac{\hbar}{i}\frac{\partial}{\partial q_2} \end{aligned}\right\} \qquad (5.1)$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the real unit vector defined by (1. 8). If we introduce the polar coordinates:

$$q_1 + iq_2 = \sqrt{\frac{\hbar}{\omega}}\,\rho e^{i\phi} \qquad (5.2)$$

the spin operator [(3. 12)] and the Hamiltonian of the system are given by

$$\left.\begin{aligned} \wedge &= \frac{1}{i}\frac{\partial}{\partial\phi} \\ \hbar\omega\boldsymbol{a}^\dagger\boldsymbol{a} &= \frac{\hbar\omega}{2}\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\partial\frac{\partial}{\partial\rho} + \frac{\wedge^2}{\rho^2} + \rho^2 - 2\right) \end{aligned}\right\} \qquad (5.3)$$

which are commutative.

We shall get the simultaneous eigenfunction of these operators. Let their eigenvalues be denoted respectively by λ and $N\hbar\omega$ [see (3. 21)]:

$$\lambda=N_l-N_r, \quad N=N_l+N_r \tag{5.4}$$

The eigenfunction is given by a linear combination of products of eigenfunctions of linear oscilator:

$$\psi_{N,\lambda}(\rho,\phi)=\sum_{n=0}^{N} c_n e^{-\frac{1}{2}\rho^2} H_n(\xi_1) H_{N-n}(\xi_2) \tag{5.5}$$

where $H_n(x)$ is the Hermite polynomial of degree n, and $\xi_i (i=1, 2)$ stand for $\sqrt{\frac{\omega}{\hbar}}q_i$ $(i=1, 2)$. The coefficient c_n is equal to the product of the matrix element of the transformation of eigenfunctions from a linear polarization to a circular polarization multiplied by a normalization factor of the former eigenfunction:

$$c_{N_1}=(\psi_{N_1,N_2}, \psi_{N_l,N_r})\frac{1}{\sqrt{2^N N_1!\,N_2!}}\sqrt{\frac{\omega}{\pi\hbar}}, \quad (N_2=N-N_1) \tag{5.6}$$

where ψ_{N_1,N_2} and ψ_{N_l,N_r} are respectively the eigenfunctions of linearly polarized and circularly polarized photons, and the parenthesis means a Hermitian nnier product. This coefficient was discussed by Pauli(6) from a standpoint of a transformation of polarization vectors. We shall calculate $\psi_{N,\lambda}$ on another point of view.

The dependence of $\psi_{N,\lambda}$ on ϕ must, due to the first equation of (5. 3), be exponential with an exponent $i\lambda\phi$. Comparing (5. 2) and (5. 5), it is easily shown that this eigenfunction must be of the form

$$\psi_{N,\lambda}(\rho,\phi)=e^{\frac{1}{2}\rho^2}e^{i\lambda\phi}\rho^{|\lambda|}u(\rho^2) \tag{5.8}$$

where $u(x)$ is a polynomial of degree $\frac{1}{2}(N-|\lambda|)$. In order that this is an eigenfunction of the Hamiltonian given by the second of (5. 3) corresponding to its eigenvalue $N\hbar\omega$, the polynomial $u(x)$ must satisfy the equation

$$\frac{d^2u}{dx^2}+\left(\frac{|\lambda|+1}{x}-1\right)\frac{du}{dx}+\frac{N-|\lambda|}{2x}u=0 \tag{5.8}$$

This equation can be obtained by differentiating $|\lambda|$-times the Laguerre equation. The polynomial $u(x)$ is therefore equal to the $|\lambda|$-th derivative of the Laguerre polynomial of degree $\frac{1}{2}(N+|\lambda|)$. Letting the norm of thus obtained eigenfunction be equal to unity, we have

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(6) W. Pauli, Geiger-Scheel's Handb. der Phys., 24/1 (1933), 252.

$$\left.\begin{aligned} u(x)&=C_{N,\lambda}\frac{d^{|\lambda|}}{dx^{|\lambda|}}L_n(x), \quad n=\frac{1}{2}(N+|\lambda|) \\ C_{N,\lambda}&=\varepsilon_{N,\lambda}\sqrt{\frac{\omega}{\pi\hbar}\frac{\frac{N-|\lambda|}{2}!}{\left(\frac{N+|\lambda|}{2}!\right)^3}} \end{aligned}\right\} \tag{5. 9}$$

where $\varepsilon_{N,\lambda}$ is a constant with its absolute value of unity.

In order to determine the factor $\varepsilon_{N,\lambda}$ we consider operations by a_l and a_r. These operators are, in the polar coordinates, given by [see (4, 13) and (3. 23)]

$$\left.\begin{aligned} a_l&=\frac{1}{2}e^{-i\phi}\left(\rho+\frac{\partial}{\partial\rho}+\frac{\Lambda}{\rho}\right) \\ a_r&=\frac{1}{2}e^{+i\phi}\left(\rho+\frac{\partial}{\partial\rho}-\frac{\Lambda}{\rho}\right) \end{aligned}\right\} \tag{5. 10}$$

Carrying out the differentiation we have

$$\left.\begin{aligned} a_l\psi_{N,\lambda}&=C_{N,\lambda}e^{i(\lambda-\quad}e^{-\frac{1}{2}\rho^2}\rho^{1-\lambda}L_n^{(1-\lambda)}(\rho^2) \quad \text{for } \lambda<0 \\ &=C_{N,\lambda}e^{(\lambda-1)\phi}e^{-\frac{1}{2}\rho^2}\rho^{\lambda-1}\left(\frac{d}{d\rho^2}\right)^{\lambda-1}n\{L'_n(\rho^2)-nL'_{n-1}(\rho^2)\} \quad \text{for} \lambda\geqq0 \end{aligned}\right\} \tag{5. 11}$$

where n stands for $\frac{1}{2}(N+|\lambda|)$. If we make use of a recurrence formula for the Laguerre polynomial, the last equation reduces to

$$a_l\psi_{N,\lambda}=-n^2C_{N,\lambda}e^{i(\lambda-1)\phi}e^{-\frac{1}{2}\rho^2}\rho^{\lambda-1}L_{n-1}^{(\lambda-1)}(\rho^2) \quad \text{for } \lambda\geqq0 \tag{5. 12}$$

The result of the operation of a_r on the eigenfunction can be obtained in the same way.

From these results we see that if we choose $\varepsilon_{N,\lambda}=(-1)^{N_l}$ in case of $\lambda\geqq0$ and $\varepsilon_{N,\lambda}=(-1)^{N_r}$ in case of $\lambda<0$, and further if we replace an explicit notation ψ_{N_l,N_r} in place of $\psi_{N,\lambda}$, the results of operations of a_l and a_r become the well known forms:

$$\begin{aligned} a_l\psi_{N_l,\,N_r} &= \sqrt{N_l}\;\psi_{N_l-1,\,N_r} \\ a_r\psi_{N_l,\,N_r} &= \sqrt{N_r}\;\psi_{N_l,\,N_r-1} \end{aligned} \tag{5.13}$$

The required normalized eigenfunction is thus given by

$$\left.\begin{aligned} \psi_{N_l,\,N_r}(q) &= (-1)^{N_l}\sqrt{\frac{\omega}{\pi\hbar}\,\frac{N_r\,!}{(N_l\,!)^3}}\,e^{i(N_l-N_r)\phi}e^{-\frac{1}{2}\rho^2}\rho^{N_l-N_r}L_{N_l}^{(N_l-N_r)}(\rho^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } N_l\geqq N_r \\ &= (-1)^{N_r}\sqrt{\frac{\omega}{\pi\hbar}\,\frac{N_l\,!}{(N_r\,!)^3}}\,e^{i(N_l-N_r)\phi}e^{-\frac{1}{2}\rho^2}\rho^{N_r-N_l}L_{N_r}^{(N_r-N_l)}(\rho^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } N_l<N_r \end{aligned}\right\} \tag{5.14}$$

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On the Correlations between Mesons and Yukawa Particles*.

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Introduction

1. At the present stage of its development, meson theory, which generated from the original idea of Yukawa, is confronted with several grave difficulties. Perhaps some of these difficulties, as has on many occasions been pointed out by Heisenberg(1), have close connections with the existence of an "universal length" that limits the validity of the present relativistic quantum mechanics. But on the other hand, it seems very likely too, that some of them are not of such essential nature, but of complicated character. In fact, it was recently shown that certain problems are of purely mathematical nature which results from an inappropriate application of perturbation theory as had been insisted on by Bhabha(2) from earlier times, and if the reaction of field is properly considered, can be removed(3)(4). The object of the present paper is to show that a sequence of difficulties can be removed by the modification of the fundamental nature of elementary particles from a new stand-point.

Initially, the Yukawa theory was proposed in order to explain the problems of nuclear forces and beta-decay phenomena in unification, and the identification of the particle introduced in this theory (Yukawa particle) with a new particle discovered in the hard component of cosmic rays (meson), at once led to theoretical stand-point from which very clear and reasonable accounts could be given for many problems on cosmic ray phenomena. For example, the theoretically anticipated instability of the Yukawa particle gave reasonable accounts for various effects on the variation of intensity of the hard component in cosmic rays (temperature effect, density effect etc.). But the more precise and quantitative the comparison between

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(*) The content of this paper was read before the symposium of the meson theory held on September, 1943. The printing was, however, delayed owing to the war circumstances.

theory and experiment became, discrepancies manifested themselves so much the more markedly. For the lifetime of mesons, experimental results exceed the theoretical by 10^2 in magnitude⑸. On the other hand, for the cross section of slow mesons with energy $2-20\times10^8$ eV, maximum estimation of experimental results⑹ gave a cross section smaller than the calculated one by 10^{-2} in magnitude. In order to remove these difficulties, many trials⑺ have been made by several authors, but they still seem unsatisfactory. In this paper, with the main aim to solve the above-mentioned two difficulties simultaneously, we propose a new theory of mesons developed from the stand-point based on the following assumption: *the meson is an elementary particle which has close correlations to the Yukawa particle, but it should be considered as an elementary particle of an different sort.*

For a meson theory from such a stand-point, two alternatives are possible: Bose meson and Fermi meson theory. In the following, we adopt the latter alternative⑻.

Interaction between Meson and Yukawa Particle.

2. According to the Yukawa theory, heavy particles (nucleon) and light particles interact with Yukawa particles by the following process (schmatically represented) respectively:

$$P \rightleftarrows N+Y^+, \qquad N \rightleftarrows P+Y^-, \qquad \text{(I)}$$

$$e^- \rightleftarrows \nu+Y^-, \qquad \nu \rightleftarrows e^-+Y^+, \qquad \text{(II)}$$

(P: proton, N: neutron, e^-: electron, ν: neutrino, $Y^\pm$: positively and negatively charged Yukawa particles).

In our theory, it is assumed that the meson is a Fermi particle with spin $\hbar/2$, and furthermore corresponding to (I) and (II) the following interactions are introduced:

$$m^\pm \rightleftarrows n+Y^\pm, \qquad n \rightleftarrows m^\pm+Y^\mp, \qquad \text{(III)}$$

($m^\pm$: positively and negatively charged meson, n: neutral meson which is assumed in the following discussions to have a negligible mass, and consequently may be regarded as equivalent with the neutrino).

Furthermore, if we introduce a neutral Yukawa particle (Y^0) in order to explain the proton-proton and neutron-neutron forces and set up the following interactions :

$$P \rightleftarrows P + Y^{\circ}, \qquad N \rightleftarrows N + Y^{\circ}, \tag{I$'$}$$

it would be natural in our theory to introduce the following interactions

$$m^{\pm} \rightleftarrows m^{\pm} + Y^{\circ}, \qquad n \rightleftarrows n + Y^{\circ}. \tag{III$'$}$$

In the following, results obtained by the introduction of new interactions (III) and (III′) will be discussed. Consequently, it is concluded that if g, g' and γ are the natural constants which represent the strength of interactions (I), (II) and (III) or (III′) respectively and we adopt as their value $g^2/\hbar c \sim 10^{-1}$, $g'^2/\hbar c \sim 10^{-15}$ and $\gamma^2/\hbar c \sim 10^{-2}$, it is possible to account for the phenomena in atomic nuclei and cosmic rays consistently, without aiming to touch the inherent difficulties of field theory.

Mass and Lifetime of Yukawa Particle.

3. For phenomena in atomic nuclei (nuclear forces, beta-decay etc.), Yukawa theory is conserved in its original form. But it is to be noted that the particle with mass determined from the range of nuclear forces, is the Yukawa particle and not the meson found in cosmic rays. This point is advantageous to explain the experimental results(9) about nuclear force range ($\hbar/m_u c$) which gives half the value ($\hbar/\mu c$) obtained from the meson mass data (m_u : mass of Yukawa particle, μ : mass of meson). In order to account for these results, $m_u = 2\mu$ is a reasonable assumption. More generally speaking, it is allowable to assume $m_u > \mu$.

4. As the consequence of the above assumption $m_u > \mu$, it becomes possible that a Yukawa particle transforms into a meson in vacuum by the following process (*spontaneous disintegration of Yukawa particle*) :

$$Y^{\pm} \rightarrow m^{\pm} + n \qquad (m_u > \mu) \tag{IV}$$

The reciprocal of the proper lifetime of Yukawa particles (t_0) calculated in vector theory is given by the following expression

$$\frac{1}{t_0} = \frac{m_u c^2}{2\hbar} \left(\frac{2}{3} \frac{\gamma_2^2}{\hbar c} + \frac{1}{3} \frac{\gamma_2^2}{\hbar c} \right) \left\{ 1 - \left(\frac{\mu}{m_u} \right)^2 \right\} \tag{1}$$

where γ_1 and γ_2 are the constants of vector and tensor interactions respectively. If we assume $\gamma_1^2 \sim 10^{-2}\,\hbar c$, $\gamma_2^2 \sim 10^{-2}\,\hbar c$ (as later shown, these figures are compatible with the considerations of meson decay), we obtain $t_0 \sim 10^{-21}$ sec. for $m_u/\mu \sim 2$. (Mean free path of Yukawa particle with the energy 10^{10} eV. is about 10^{-8} cm). Naturally these values depend on the ratio m_u/μ. As m_u approaches μ, the lifetime of Yukawa particles is prolonged by the last factor in (1) ; and the limit $m_u = \mu$, it becomes infinite. But in this case the Yukawa particle decays by the original process $Y \rightarrow e + \nu$. Consequently, the lifetime of the Yukawa particle does not become greater than 10^{-8} sec.

5. Process (IV) represents the creation of the hard component in cosmic rays. For the analysis of cosmic ray phenomena, we shall take up proton primary hypothesis which has been proposed by Schein et al.(10). Then primary incidental protons at first create Yukawa particles by collisions with nuclei of N or O atoms existiug in the atmosphere. These Yukawa particles transform into mesons instantaneously by the above process. The interaction of the latter with matter is smaller than that of the former ($\gamma < g$). Thus we observe these mesons as hard components of cosmic-rays. Previously, Nordheim(11) in his analysis on cosmic-ray suggested that the absorption process of hard component must have cross sections smaller by a factor of order 10 compared to the creation processes. This difficulty, which indicates an asymmetry inconsistent with the original Yukawa theory, is overhelmed by the insertion of process (IV).

Decay and Scattering of Meson.

6. According to our theory, the decay of mesons occurs by following process :

$$\begin{tikzcd}
 & Y^{\pm} + n \arrow[dr] & \\
m^{\pm} \arrow[ur] \arrow[dr] & & e^{\pm} + \nu + n. \\
 & m^{\pm} + e^{\pm} + \nu + Y^{\mp} \arrow[ur] &
\end{tikzcd} \tag{V}$$

The reciprocal of the proper lifetime of mesons (τ_0) in vector and pseudoscalar theory are given as follows, where f', $\bar{\gamma}$ are interaction constants in pseudoscalar theory corresponding to g', γ in vector theory.

$$\frac{1}{\tau_0} \doteq \frac{\mu c^2}{\pi \hbar} \left\{ \left(\frac{g'_1 \gamma_1}{\hbar c} \right) I_1^{(v)} + \left(\frac{g'_2 \gamma_2}{\hbar c} \right) I_2^{(v)} \right\}, \qquad (2) \quad \text{(vector theory)}$$

$$\frac{1}{\tau_0} \doteq \frac{\mu c^2}{\pi \hbar} \left\{ \left(\frac{f'_1 \bar{\gamma}_1}{\hbar c} \right) I_1^{(p)} + \left(\frac{f'_2 \bar{\gamma}_2}{\hbar c} \right) I_2^{(p)} \right\}, \qquad (2') \quad \text{(pseudoscalar theory)}$$

$$\left.\begin{aligned}
I_1^{(v)}&=\frac{1}{(1+A)^2}\left\{\frac{300\,A^3+870\,A^2+670\,A+603}{1440}-\frac{A(A+1)(5A^2+12A+6)}{24}\times\log\frac{A+1}{A}\right\},\\
I_2^{(v)}&=\frac{1}{(1+A)^2}\left\{\frac{30\,A^3+285\,A^2+310\,A+61}{1446}-\frac{A(A+1)(A^2+9A+6)}{48}\times\log\frac{A+1}{A}\right\},\\
I_1^{(p)}&=\frac{6\,A+1}{16}-\frac{A(3\,A+2)}{8}\log\frac{A+1}{A}\qquad I_2^{(p)}=\frac{1}{(1+A)^2}\frac{1}{480},\\
A&=\left(\frac{m_u}{\mu}\right)^2-1.
\end{aligned}\right\}\quad(3)$$

The value of τ_0 determined from experiments is about 10^{-6} sec. Using this value and characteristic constants of beta-radioactive nuclei, which depend on constants g' or f', we can determine the value of interaction constants γ or $\bar{\gamma}$. Using only the first term in (2), we can determine $\gamma_1^2\sim0.025\,\hbar c$ for $m_u/\mu\sim2$. Determining γ_2 from the second term only, we obtain a magnitude of the same order.

7. Introducing the neutral Yukawa particle and taking interaction (III′), the scattering of mesons occurs by the following process:

$$m^+ + N \rightarrow m^+ + Y^\circ + N \rightarrow m^{+\prime} + N' \qquad \text{(VI)}$$

The cross section of these processes is determined by the interaction constant of (III′). Taking the same constant γ of (III) for this (analogous to Kemmer's symmetrical theory of nuclear forces), we have to examine whether the above-determined γ leads to a cross section consistent with scattering experiments or not. Furthermore, we must take into account the following process which has a probability of the same order of magnitude as that of the scattering process.

$$m^+ + N \rightarrow m^+ + Y^- + P \rightarrow n + P \qquad \text{(VI}'\text{)}$$

If the cross section of the latter process becomes appreciably large, it is provable that an attempt to account for the creation of neutral mesons by this process may lead to a contradiction with experimental results. From these considerations, it is desirable to take as small a value for γ as possible(*)(**). In fact, taking into account that expressions (2) and (2′) consist of two terms which involve g_1', g_2', (or f_1', f_2') separately, and adjusting their values properly, we can make the values of γ's sufficiently small. For example, in the vector theory if we assume $g_1=0$, then g_1' becomes irrelevant to the β-decay process and we can choose a sufficiently small value for γ_1. Furthermore, if it is assumed that meson decay occurs by γ_1-interaction only, the choice of γ_2 can be determined from the scattering cross section freely. In pseudo-scalar theory, if we take into consideration that f_1, f_1' are not responsible for nuclear force and β-decay process respectively, we can make $\bar{\gamma}_1$, $\bar{\gamma}_2$ sufficiently small simultaneously in an analogous way.

In the above consideration we took up the symmetrical formalism. But even if we assume that for the interaction with heavy particles the interaction constants of Y° are greater than that of $Y^\pm$ as in Heitler's theory or Bethe's neutral theory, it is still possible to make γ_1, γ_2 sufficiently small.

Some Remarks on the Results.

8. As shown above, there is certainly a possibility that the contradiction between Yukawa theory and experimental results be removed. In the following, some related problems which are characteristic to our theory will be discussed.

(i) One of characteristics of our theory is that the meson has spin $\hbar/2$. This point is interesting in relation with the results of analysis on radiative effects of the meson which are sensitively affected by the spin value. According to the calculation by Christy and Kusaka(12) of the sizes of bursts produced by mesons, our choice of spin $\hbar/2$ for the meson is supported by experiment, in contrast to the usual theory with spin $\hbar$.

(ii) The rôle of neutral particles in the original Yukawa theory has been frequently discussed. If we take the symmetrical theory on nuclear force and furthermore assume $m_u > 2\mu$, the neutral Yukawa particle decays

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(*) Naturally, we cannot take too small a value for γ, in order that it be consistent with the decay probability of Yukawa particles. Eventually, it seems appropriate to take $\gamma^2 \sim 10^{-2}\,\hbar c$ (or a slightly smaller value).

(**) If we take the formalism which excludes the neutral Yukawa particle, the scattering of mesons takes place only by a process of the fourth order, which results in a much smaller cross section than that of (VI). But, in this case, process (VI′) is not excluded.

by the following process: (*spontaneous disintegration of neutral Yukawa particle*)

$$Y^0 \begin{array}{l} \nearrow m^+ + m^- \\ \searrow n + n \end{array} \tag{V'}$$

and its proper lifetime is of the same order as that of charged Yukawa particles calculated above ($\sim 10^{-21}$ sec). This value is smaller than that obtained from the previously discussed process(13), in which the neutral Yukawa particle (neutretto) transforms into photons. As a result, the Yukawa particle transforms into mesons (penetrating component) with a greater probability than into photons (shower producing component). This is very favourable in interpreting the results of experiment by Schein et al(10) which denied the existence of shower producing components in upper atmosphere and suggested, as the possible alternatives which was at first suggested by Taketani(14), the introduction of some process in which the neutral Yukawa particle transforms into the penetrating component with a greater probability than into shower-producing components, or otherwise the elimination of neutral Yukawa particle from the formalism.

(iii) Also, it is to be noted that as the decay product of one meson, three particles are generated and two of them are neutral particles which have smaller interaction with matter. This consequence seems to be in agreement with the fact(15) that when mesons are stopped in matter, decayed electrons have rarely been observed, or that when mesons decay in a Wilson chamber, only slow electrons have been observed. Furthermore, the fraction of the total mesonic energy given to the electron by decay in this theory is smaller than that in the usual theory. This point is, too, supported by Nordheim's analysis(16) on the intensity of cascade showers. But eventually, the validity of introduction of such processes should be fully discussed in future, based on further experimental and theoretical investigations on cosmic rays.

Finally, the authors wish to express their appreciation to Prof. H. Yukawa for his continuous interest in this work, and to Y. Tanikawa and S. Nakamura for constant collaboration. They are also much indebted to M. Taketani for his valuable discussion.

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Errata: On the Generalized Transformation Functions.

By Y. Tanikawa.

[Prog. Theor. Phys. 1, (1946), 12].

Page 13, line 9 and 10, for $q_t \equiv (q_{1t}, q_{2t}, \cdots\cdots q_{rt})$, $q_T \equiv (q_{1T}, q_{2T}, \cdots\cdots q_{rT})$, read $q_t \equiv (q_{1t}, q_{2t}, \cdots\cdots q_{rt})$, $q_T \equiv (q_{1T}, q_{2T}, \cdots\cdots q_{rT})$.

Page 14, line 14, for $\tilde{M}q' = Mq'$, $Mq'^2 = Mq'$ read $\tilde{M}q = Mq$, $(Mq)^2 = Mq$.

For the symbols Mq, Mq', Mq'' which are found in this and later pages, q, q', q'' should be refered as *suffices* to M, although those are represented in relatively large scale.

Page 16, line 3, for q_t's, q_T''s read q_t's and q_T's.

Page 16, line 30, for *sufaee* read *surfacc*.

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